Vectors and Vector Spaces

Table of Contents

Interactive Visualizations

HTML Interactive Demos

Chapter 1 Interactive Hub - Navigate all visualizations for this chapter

Individual Lesson Visuals:

- **Vector Basics** - Magnitude & direction fundamentals
- **Vector Operations** - Addition, scaling, dot products with cosine relationships
- **Linear Independence** - Mathematical analysis of independence
- **Abstract Spaces** - Beyond arrows: polynomials, functions, matrices

Complete Experiences:

- **Interactive Demo Suite** - Complete Python visualization suite in HTML format

---

Introduction to Vectors

A **vector** is a mathematical object that has both magnitude and direction. In $\mathbb{R}^n$, a vector can be represented as an ordered list of n real numbers:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Geometric Interpretation

- In $\mathbb{R}^2$: vectors represent points in a plane or arrows from origin
- In $\mathbb{R}^3$: vectors represent points in 3D space
- In $\mathbb{R}^n$: vectors represent points in n-dimensional space

Examples

$$\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$$

Interactive: Explore vector fundamentals with **Vector Basics** - see how magnitude and direction define everything about a vector.

Vector Operations

Vector Addition

For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

Properties:

- **Commutative:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- **Associative:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- **Identity:** $\mathbf{v} + \mathbf{0} = \mathbf{v}$

Scalar Multiplication

For scalar $c \in \mathbb{R}$ and vector $\mathbf{v} \in \mathbb{R}^n$:

$$c\mathbf{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}$$

Properties:

- **Distributive:** $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- **Distributive:** $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$
- **Associative:** $(cd)\mathbf{v} = c(d\mathbf{v})$
- **Identity:** $1\mathbf{v} = \mathbf{v}$

Dot Product (Inner Product)

For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Properties:

- **Commutative:** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- **Distributive:** $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- **Homogeneous:** $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- **Positive definite:** $\mathbf{v} \cdot \mathbf{v} \geq 0$, with equality iff $\mathbf{v} = \mathbf{0}$

Vector Magnitude (Norm)

The **magnitude** or **norm** of vector $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Unit Vectors

A **unit vector** has magnitude 1. Any non-zero vector $\mathbf{v}$ can be normalized:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Interactive: Practice vector operations with **Vector Operations** - see addition, scaling, dot products, and the cosine relationship in action.

Vector Spaces

A **vector space** V over field $\mathbb{F}$ (usually $\mathbb{R}$ or $\mathbb{C}$) is a set with two operations:

> **Field Definition:** A **field** $\mathbb{F}$ is a set equipped with two operations (addition and multiplication) that satisfy the usual arithmetic properties: commutativity, associativity, distributivity, existence of additive and multiplicative identities (0 and 1), and existence of additive inverses for all elements and multiplicative inverses for all non-zero elements. Common examples include the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and rational numbers $\mathbb{Q}$. - Vector addition: $+ : V \times V \to V$ - Scalar multiplication: $\cdot : \mathbb{F} \times V \to V$

Vector Space Axioms

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $a, b \in \mathbb{F}$:

1. **Closure under addition**: $\mathbf{u} + \mathbf{v} \in V$
2. **Commutativity**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Associativity**: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. **Zero vector**: $\exists \mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
5. **Additive inverse**: $\forall \mathbf{v} \in V, \exists(-\mathbf{v}) \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
6. **Closure under scalar multiplication**: $a\mathbf{v} \in V$
7. **Scalar multiplication identity**: $1\mathbf{v} = \mathbf{v}$
8. **Scalar multiplication associativity**: $(ab)\mathbf{v} = a(b\mathbf{v})$
9. **Distributivity**: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
10. **Distributivity**: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Examples of Vector Spaces

- $\mathbb{R}^n$: n-dimensional real coordinate space
- $\mathbb{C}^n$: n-dimensional complex coordinate space
- P_n: polynomials of degree at most n
- $C[a, b]$: continuous functions on interval $[a, b]$
- $M_{m \times n}$: $m \times n$ matrices

Concrete examples you can picture

- $\mathbb{R}^2$: all 2D arrows from the origin. Examples: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 0.5 \end{pmatrix}$. Under addition and scalar multiplication you stay in $\mathbb{R}^2$.
- $\mathbb{R}^3$: all 3D arrows. Examples: $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$.
- $\mathbb{C}^2$: complex 2-vectors. Example: $\begin{pmatrix} 1 + i \\ -2i \end{pmatrix}$.
- P_2: all polynomials of degree ≤ 2. General element: $a_0 + a_1 x + a_2 x^2$. Concrete examples: $1 - 3x + 2x^2$, $\frac{1}{2} + x$.

- $C[0,1]$: continuous real-valued functions on $[0,1]$. Examples: $f(x) = \sin x$, $g(x) = x^2$, $h(x) = e^x$ (restricted to $[0,1]$).
- $M_{2\times 2}$: all 2×2 real matrices. Example: $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.
- **Subspaces of** $\mathbb{R}^n$: any line or plane through the origin. Example in $\mathbb{R}^2$: all multiples of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; in $\mathbb{R}^3$: the xy-plane $\{(x, y, 0)\}$.

  This is essentially saying that a vector space is any collection of "vectors" (not necessarily geometric arrows) where adding two of them and scaling by numbers keeps you inside the same collection, and the usual algebraic rules hold.

Interactive: Discover abstract vector spaces with **Abstract Vector Spaces** - see how polynomials, functions, and matrices can all be "vectors".

Linear Independence

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is **linearly independent** if the only solution to:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_k = 0$.

Linear Dependence

Vectors are **linearly dependent** if there exist scalars $c_1, c_2, \dots, c_k$ (not all zero) such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

Key Properties

- **Zero vector dependency:** Any set containing the zero vector is linearly dependent
- **Two-vector dependency:** A set of two vectors is linearly dependent iff one is a scalar multiple of the other
- **Dimension limit:** In $\mathbb{R}^n$, any set of more than n vectors is linearly dependent

Visual intuition and concrete examples (in $\mathbb{R}^2$)

- **Linearly independent example.** Take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. They point in two different directions and span the whole plane. The only way to make $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ is $c_1 = c_2 = 0$.
- **Linearly dependent example.** Take $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{w}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\mathbf{w}_1$. They lie on the same line through the origin (no new direction). Here $2\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{0}$ with nonzero coefficients, so they are dependent.

  Intuitively: linear independence means "each vector brings a new direction you couldn't already get by mixing the others". Dependence means "at least one vector is redundant because it can be built from the rest".

Quick mental picture

- Two independent vectors in $\mathbb{R}^2$ form a grid (a parallelogram basis). Two dependent vectors line up on the same straight line.

Interactive: Explore independence concepts with **Linear Independence** - see when vectors add genuine new possibilities versus when they're just echoes.

Basis and Dimension

Span

The **span** of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_i \in \mathbb{F}\}$$

Intuition: The span of a set of vectors = all possible linear combinations of them. Think of it as the space you can reach by scaling and adding those vectors.

Geometric illustrations (use arrows for vectors and label the spaces): - In $\mathbb{R}^2$: - span of one nonzero vector $\vec{v} \to$ a line through the origin along $\vec{v}$ - span of two independent vectors $\vec{u}, \vec{v} \to$ the entire plane $\mathbb{R}^2$ - In $\mathbb{R}^3$: - span of one nonzero vector $\vec{v} \to$ a line through the origin - span of two independent vectors $\vec{u}, \vec{v} \to$ a plane through the origin - span of three independent vectors $\vec{u}, \vec{v}, \vec{w} \to$ the whole space $\mathbb{R}^3$

Real-life analogy: Span is like the territory covered when you combine directions. The pair {east, north} lets you reach any place on a flat floor; one single direction only gives a line of reachable points.

Basis

A **basis** for vector space V is a set of vectors that: 1. Spans V 2. Is linearly independent

Intuition: A basis is the simplest set of building blocks of the space. It is the smallest independent set of vectors that covers the whole space. If any vector in the set could be built from the others, it would be redundant — and the set would not be a basis.

Consistency note for diagrams: we draw vectors as arrows (e.g., $\vec{u}, \vec{v}$) and label the resulting space (line, plane, or whole space) accordingly.

> Key rule (highlight): If $\dim(V) = n$, then any linearly independent set of n vectors is a basis, and any spanning set of n vectors is also a basis.

Standard Basis for $\mathbb{R}^n$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Dimension

The **dimension** of vector space V is the number of vectors in any basis for V.

$$\dim(V) = \text{number of vectors in basis}$$

Important Theorems

1. **Basis Theorem**: All bases of a finite-dimensional vector space have the same number of vectors
2. **Dimension Theorem**: If $\dim(V) = n$, then:
   - Any linearly independent set of n vectors is a basis
   - Any spanning set of n vectors is a basis
   - Any linearly independent set has at most n vectors
   - Any spanning set has at least n vectors

Practice Problems

Problem 1

Given vectors $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$:

a) Calculate $\mathbf{u} + \mathbf{v}$
b) Calculate $3\mathbf{u} - 2\mathbf{v}$
c) Calculate $\mathbf{u} \cdot \mathbf{v}$
d) Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$

Problem 2

Determine if the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ are linearly independent.

Problem 3

Find a basis for the subspace spanned by:

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

Next: Matrix Operations and Properties

Solutions

Solution to Problem 1

Given $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$:

- $\mathbf{u} + \mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$
- $3\mathbf{u} - 2\mathbf{v} = 3\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ 13 \end{pmatrix}$
- $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + (-1) \cdot 4 + 3 \cdot (-2) = -8$
- $\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$, $\|\mathbf{v}\| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{21}$

Solution to Problem 2

Let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$. With $\mathbf{v}_1 = (1, 2, 1)^T$, $\mathbf{v}_2 = (2, 1, 3)^T$, $\mathbf{v}_3 = (1, -1, 2)^T$.

Compute $\det A = 0$, so the set is linearly dependent. In fact, $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$, i.e., $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ with nonzero coefficients.

Solution to Problem 3

Given $\mathbf{w}_1 = (1, 2, 1, 0)^T$, $\mathbf{w}_2 = (2, 1, 0, 1)^T$, $\mathbf{w}_3 = (3, 3, 1, 1)^T$.

Observe $\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_2$. A basis for $\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is therefore $\{\mathbf{w}_1, \mathbf{w}_2\}$ (dimension 2).