Matrices and Matrix Operations

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Introduction to Matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. An $m \times n$ matrix has m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Notation

- A_{ij} or a_{ij}: element in row i, column j
- $A \in \mathbb{R}^{m\times n}$: A is an $m \times n$ real matrix
- $A \in \mathbb{C}^{m\times n}$: A is an $m \times n$ complex matrix

Examples

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 4 \end{pmatrix}, \quad C = (5)$$

What does each entry in a matrix mean?

The meaning depends on context. A matrix is a rectangular array of numbers, but its entries can represent very different things:

1. **Data Table (rows = objects, columns = features).** Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

   Row 1 could be student #1's scores on 3 exams; row 2 student #2's scores.

2. **Linear Transformation (mapping vectors from one space to another).** Example:

$$B = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 4 \end{pmatrix}$$

   - The first column $(2, 0, 1)^T$ is where $(1, 0)^T$ (the x-axis) gets mapped.
   - The second column $(-1, 3, 4)^T$ is where $(0, 1)^T$ (the y-axis) gets mapped. Multiplying Bx shows how a vector in the plane is transformed into $\mathbb{R}^3$.

3. **Graph (Adjacency Matrix).** For a graph with 3 nodes:

$$G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   - Entry $g_{12} = 1$ means an edge between node 1 and 2.
   - Zeros on the diagonal mean no self-loops. Thus, matrices can encode relationships and networks.

Consistency note: Keep arrows for vectors and label rows/columns when helpful.

Interactive: Explore what matrices represent with **Matrix Introduction** - see how the same mathematical object can be data, transformations, or networks.

Matrix Operations

Matrix Addition

For matrices $A, B \in \mathbb{R}^{m\times n}$:

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Scalar Multiplication

For scalar $c \in \mathbb{R}$ and matrix $A \in \mathbb{R}^{m\times n}$:

$$(cA)_{ij} = c \cdot A_{ij}$$

Example:

$$3\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$$

Matrix Multiplication

For $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{p\times n}$, the product $AB \in \mathbb{R}^{m\times n}$:

$$(AB)_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}$$

Key Points:

- **Dimension compatibility:** Number of columns in A must equal number of rows in B
- **Not commutative:** Matrix multiplication is **not commutative**: $AB \neq BA$ (in general)
- **Associative:** Matrix multiplication is **associative**: $(AB)C = A(BC)$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Interactive: Master matrix operations with **Matrix Operations** - see how addition combines effects, multiplication composes transformations, and transpose changes perspective.

Matrix Transpose

The **transpose** of matrix $A \in \mathbb{R}^{m\times n}$ is $A^T \in \mathbb{R}^{n\times m}$:

$$(A^T)_{ij} = A_{ji}$$

Properties:

- **Double transpose:** $(A^T)^T = A$
- **Addition transpose:** $(A+B)^T = A^T + B^T$
- **Scalar transpose:** $(cA)^T = cA^T$
- **Product transpose:** $(AB)^T = B^T A^T$

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Special Types of Matrices

Square Matrices

A matrix $A \in \mathbb{R}^{n\times n}$ where number of rows equals number of columns.

Identity Matrix

The $n \times n$ **identity matrix** I_n has 1's on the diagonal and 0's elsewhere:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Property: $AI = IA = A$ for any compatible matrix A.

Zero Matrix

The **zero matrix** O has all entries equal to 0:

$$O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Diagonal Matrix

A square matrix where all non-diagonal entries are zero:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

Symmetric Matrix

A square matrix A is **symmetric** if $A = A^T$:

$$A_{ij} = A_{ji} \text{ for all } i, j$$

Skew-Symmetric Matrix

A square matrix A is **skew-symmetric** if $A = -A^T$:

$$A_{ij} = -A_{ji} \text{ for all } i, j$$

Upper/Lower Triangular Matrices

- **Upper triangular**: $A_{ij} = 0$ for $i > j$
- **Lower triangular**: $A_{ij} = 0$ for $i < j$

Examples:

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

Interactive: Meet the matrix personalities with **Special Matrices -** see how identity preserves, diagonal stretches, symmetric reflects, and more.

Matrix Properties

Trace

The **trace** of square matrix $A \in \mathbb{R}^{n\times n}$:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii} = A_{11} + A_{22} + \cdots + A_{nn}$$

Properties:

- **Linearity:** $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
- **Scalar multiplication:** $\operatorname{tr}(cA) = c \cdot \operatorname{tr}(A)$
- **Cyclic property:** $\operatorname{tr}(AB) = \operatorname{tr}(BA)$
- **Transpose invariance:** $\operatorname{tr}(A^T) = \operatorname{tr}(A)$

Matrix Inverse

For square matrix $A \in \mathbb{R}^{n\times n}$, the **inverse** A^{-1} satisfies:

$$AA^{-1} = A^{-1}A = I$$

Properties:

- **Double inverse:** $(A^{-1})^{-1} = A$
- **Product inverse:** $(AB)^{-1} = B^{-1}A^{-1}$
- **Transpose inverse:** $(A^T)^{-1} = (A^{-1})^T$
- **Determinant inverse:** $\det(A^{-1}) = \frac{1}{\det(A)}$

Determinant

For square matrix $A \in \mathbb{R}^{n\times n}$, the **determinant** $\det(A)$ is a scalar.

For 2×2 matrices:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

For 3×3 matrices (cofactor expansion):

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Properties:

- **Multiplicativity:** $\det(AB) = \det(A)\det(B)$
- **Transpose invariance:** $\det(A^T) = \det(A)$
- **Scalar scaling:** $\det(cA) = c^n \det(A)$ for $n \times n$ matrix
- **Invertibility criterion:** A is invertible iff $\det(A) \neq 0$

Interactive: Understand geometric invariants with **Determinant & Trace** - see how determinant measures area scaling and trace connects to eigenvalues.

Systems of Linear Equations

A system of m linear equations in n unknowns can be written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Matrix Form

$$A\mathbf{x} = \mathbf{b}$$

where:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Solution Methods

1. Matrix Inverse (when A is square and invertible)

$$\mathbf{x} = A^{-1}\mathbf{b}$$

2. Gaussian Elimination Transform the augmented matrix $[A|\mathbf{b}]$ to row echelon form.

3. Cramer's Rule (for square systems with $\det(A) \neq 0$)

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is A with column i replaced by $\mathbf{b}$.

Solution Types

1. **Unique solution**: $\det(A) \neq 0$ (for square systems)
2. **No solution**: System is inconsistent
3. **Infinitely many solutions**: System is underdetermined

Practice Problems

Problem 1

Given matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Calculate: a) $A + B$ b) AB c) BA d) A^T e) $\det(A)$

Problem 2

Find the inverse of:

$$C = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

Problem 3

Solve the system using matrix methods:

$$\begin{aligned} 2x + y &= 5 \\ x + 3y &= 7 \end{aligned}$$

Problem 4

For what values of k does the following system have: a) A unique solution b) No solution c) Infinitely many solutions

$$\begin{aligned} x + 2y &= 3 \\ 2x + ky &= 6 \end{aligned}$$

Next: Linear Transformations

Solutions

Solution to Problem 1

Given $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$:

- a) $A + B = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$

- b) $AB = \begin{pmatrix} 2\cdot 1 + 1\cdot 0 & 2\cdot 2 + 1\cdot 1 \\ 3\cdot 1 + 4\cdot 0 & 3\cdot 2 + 4\cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 10 \end{pmatrix}$
- c) $BA = \begin{pmatrix} 1\cdot 2 + 2\cdot 3 & 1\cdot 1 + 2\cdot 4 \\ 0\cdot 2 + 1\cdot 3 & 0\cdot 1 + 1\cdot 4 \end{pmatrix} = \begin{pmatrix} 8 & 9 \\ 3 & 4 \end{pmatrix}$
- d) $A^T = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$
- e) $\det(A) = 2\cdot 4 - 1\cdot 3 = 5$

Solution to Problem 2

$C = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, $\det C = 3\cdot 1 - 1\cdot 2 = 1$. Hence

$$C^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

Solution to Problem 3

Solve $\begin{cases} 2x + y = 5 \\ x + 3y = 7 \end{cases}$. From the first, $y = 5 - 2x$. Substitute: $x + 3(5 - 2x) = 7 \Rightarrow x + 15 - 6x = 7 \Rightarrow -5x = -8 \Rightarrow x = \frac{8}{5}$. Then $y = 5 - \frac{16}{5} = \frac{9}{5}$.

Solution to Problem 4

System $\begin{cases} x + 2y = 3 \\ 2x + ky = 6 \end{cases}$ has coefficient matrix $\begin{pmatrix} 1 & 2 \\ 2 & k \end{pmatrix}$ with determinant $k - 4$.

- a) Unique solution iff $k \neq 4$.
- b) No solution when $k = 4$ and RHS inconsistent. Here second equation equals $2(x + 2y) = 6$, which is consistent with first, so "no solution" never occurs; it gives infinitely many.
- c) Infinitely many solutions when $k = 4$ (dependent equations).