Linear Transformations

Table of Contents

Definition of Linear Transformations

A **linear transformation** (or linear map) $T : V \to W$ between vector spaces V and W satisfies:

1. **Additivity**: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$
2. **Homogeneity**: $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $c \in \mathbb{F}$, $\mathbf{v} \in V$

Equivalent Condition

T is linear iff $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all scalars c_1, c_2 and vectors $\mathbf{v}_1, \mathbf{v}_2$.

Important Properties

- $T(\mathbf{0}) = \mathbf{0}$ (zero vector maps to zero vector)
- $T(-\mathbf{v}) = -T(\mathbf{v})$
- $T\left(\sum_{i=1}^{n} c_i\mathbf{v}_i\right) = \sum_{i=1}^{n} c_iT(\mathbf{v}_i)$

Matrix Representation

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by an $m \times n$ matrix A such that:

$$T(\mathbf{x}) = A\mathbf{x}$$

Finding the Matrix Representation

If $T : \mathbb{R}^n \to \mathbb{R}^m$, then the matrix A has columns:

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)]$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$.

Example

Find the matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x - y \\ 3x \end{pmatrix}$$

Solution:

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix},\quad T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

Therefore: $A=\begin{pmatrix}2&1\\1&-1\\3&0\end{pmatrix}$

Common Linear Transformations

1. Scaling (Dilation)

Scale by factor k in all directions:

$$T(\mathbf{x})=k\mathbf{x},\quad A=kI=\begin{pmatrix}k&0\\0&k\end{pmatrix}$$

2. Reflection

Reflection across x-axis:

$$A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Reflection across y-axis:

$$A=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

Reflection across line $y=x$:

$$A=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

3. Rotation

Counterclockwise rotation by angle θ:

$$A=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$$

Common angles: - 90ř: $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ - 180ř: $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ - 270ř: $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

4. Shearing

Horizontal shear:

$$A=\begin{pmatrix}1&k\\0&1\end{pmatrix}$$

Vertical shear:

$$A=\begin{pmatrix}1&0\\k&1\end{pmatrix}$$

5. Projection

Orthogonal projection onto x-axis:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Orthogonal projection onto line through origin with direction u:

$$P = \frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}}$$

> Note: This formula requires $\mathbf{u} \neq \mathbf{0}$. If you normalize $\mathbf{u}$ to a unit vector $\hat{\mathbf{u}} = \mathbf{u}/\|\mathbf{u}\|$, then the projection matrix simplifies to $P = \hat{\mathbf{u}}\,\hat{\mathbf{u}}^T$.

Properties of Linear Transformations

Rank and Nullity

For linear transformation $T : V \to W$: - **Rank**: $\text{rank}(T) = \dim(\text{Im}(T))$ - **Nullity**: $\text{nullity}(T) = \dim(\text{Ker}(T))$

Rank-Nullity Theorem

$$\dim(V) = \text{rank}(T) + \text{nullity}(T)$$

Types of Linear Transformations

Injective (One-to-One) T is **injective** if $T(\mathbf{u}) = T(\mathbf{v}) \Rightarrow \mathbf{u} = \mathbf{v}$

Equivalent conditions: - $\text{Ker}(T) = \{\mathbf{0}\}$ - $\text{nullity}(T) = 0$ - Columns of matrix A are linearly independent

Surjective (Onto) T is **surjective** if for every $\mathbf{w} \in W$, there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$

Equivalent conditions: - $\text{Im}(T) = W$ - $\text{rank}(T) = \dim(W)$ - Columns of matrix A span $\mathbb{R}^m$

Bijective (Isomorphism) T is **bijective** if it's both injective and surjective.

Equivalent conditions: - T has an inverse T^{-1} - Matrix A is invertible (square and $\det(A) \neq 0$)

Kernel and Image

Kernel (Null Space)

The **kernel** of $T : V \to W$ is:

$$\text{Ker}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$$

For matrix A, this is the **null space**:

$$\text{Null}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

Image (Range/Column Space)

The **image** of $T : V \to W$ is:

$$\text{Im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

For matrix A, this is the **column space**:

$$\text{Col}(A) = \text{span}\{\text{columns of } A\}$$

Finding Kernel and Image

Example: Find kernel and image of $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$

Kernel: Solve $A\mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Row reduction gives: $\text{Ker}(A) = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$

Image: Find basis for column space

$$\text{Im}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\right\}$$

Composition and Inverse

Composition of Linear Transformations

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear, then $(T_2 \circ T_1) : U \to W$ is linear:

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$$

Matrix representation: If T_1 has matrix A_1 and T_2 has matrix A_2, then $T_2 \circ T_1$ has matrix $A_2 A_1$.

Inverse Linear Transformation

If $T : V \to W$ is bijective, then $T^{-1} : W \to V$ exists and is linear.

Matrix representation: If T has matrix A, then T^{-1} has matrix A^{-1}.

Change of Basis

If $T : V \to V$ and we change from basis $\mathcal{B}$ to basis $\mathcal{C}$, the matrix changes:

$$[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$$

where P is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

Practice Problems

Problem 1

Determine which of the following are linear transformations:

a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ y \end{pmatrix}$

c) $T : \mathbb{R}^2 \to \mathbb{R}$, $T \begin{pmatrix} x \\ y \end{pmatrix} = 3x - 2y + 1$

Problem 2

Find the matrix representation of the linear transformation that: - Rotates by 45ř counterclockwise - Then reflects across the x-axis

Problem 3

For $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$:

a) Find $\text{Ker}(A)$
b) Find $\text{Im}(A)$
c) Verify the rank-nullity theorem

Problem 4

Given $T : \mathbb{R}^3 \to \mathbb{R}^2$ with matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$:

a) Is T injective?
b) Is T surjective?
c) Find a vector in $\text{Ker}(T)$ if it exists

Next: Eigenvalues and Eigenvectors

Solutions

Solution to Problem 1

a) Linear (additivity and homogeneity both hold).
b) Not linear (contains x^2).
c) Not linear (constant term $+1$ violates $T(0) = 0$ and additivity).

Solution to Problem 2

Rotation by 45° then reflect across x-axis: $A = R_{45}\, R_{\text{x-ref}}$ or, applying reflection after rotation, $A = R_{\text{x-ref}}\, R_{45}$ depending on stated order. With reflection after rotation:

$$R_{45} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad R_{\text{x-ref}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = R_{\text{x-ref}} R_{45} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Solution to Problem 3

For $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}$ (from the example): Row reduction shows $\operatorname{rank}(A) = 2$. - Kernel: Solve $A\mathbf{x} = 0$ gives $\mathbf{x} = t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. So $\operatorname{Ker}(A) = \operatorname{span}\{(2, -1, 1)^T\}$. - Image/column space: Columns satisfy $c_1 = c_2 - c_3$, so a basis is $\{c_2, c_3\} = \{(2, 1, 2)^T, (1, 3, 2)^T\}$.

Solution to Problem 4

Composition corresponds to matrix multiplication. If T_1 has A_1 and T_2 has A_2, then $(T_2 \circ T_1)$ has matrix $A_2 A_1$ (order matters).