Eigenvalues and Eigenvectors

Table of Contents

Definition and Basic Concepts

Eigenvalues and Eigenvectors

For square matrix $A \in \mathbb{R}^{n\times n}$: - **Eigenvector**: A non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ - **Eigenvalue**: The scalar λ corresponding to eigenvector $\mathbf{v}$

Geometric Interpretation

An eigenvector is a direction that remains unchanged (up to scaling) under the linear transformation represented by A.

> Intuition: A stretches or squashes space differently in different directions. Along certain special directions (the eigenvectors), the action is a pure stretch by a factor λ (possibly a flip if $\lambda < 0$). This is why eigenvalues show up in applications like PCA (principal directions of variance) and stability of dynamical systems (growth/decay rates along invariant directions).

Eigenspace

The **eigenspace** E_λ corresponding to eigenvalue λ is:

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\} = \mathrm{Ker}(A - \lambda I)$$

Finding Eigenvalues and Eigenvectors

Step 1: Find Eigenvalues

Solve the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

The **characteristic polynomial** is:

$$p(\lambda) = \det(A - \lambda I)$$

Step 2: Find Eigenvectors

For each eigenvalue λ_i, solve:

$$(A - \lambda_i I)\mathbf{v} = \mathbf{0}$$

Example: 2×2 Matrix

Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

Step 1: Characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) = 0$$

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2$

Step 2: Eigenvectors

For $\lambda_1 = 3$:

$$(A - 3I)\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 2$:

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector: $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Animation: Animation: Watch "Eigenvalues & Eigenvectors" demo to see how eigenvectors remain on the same line while being scaled by their eigenvalues.

Properties of Eigenvalues and Eigenvectors

Basic Properties

1. **Trace**: $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$ (sum of eigenvalues)
2. **Determinant**: $\det(A) = \prod_{i=1}^{n} \lambda_i$ (product of eigenvalues)
3. **Zero eigenvalue**: $\lambda = 0$ is an eigenvalue iff A is singular
4. **Eigenvectors**: Eigenvectors corresponding to different eigenvalues are linearly independent

Algebraic and Geometric Multiplicity

- **Algebraic multiplicity**: Multiplicity of λ as a root of characteristic polynomial
- **Geometric multiplicity**: $\dim(E_\lambda) = \text{nullity}(A - \lambda I)$

Important: Geometric multiplicity $\leq$ Algebraic multiplicity

Similar Matrices

Matrices A and B are **similar** if $B = P^{-1}AP$ for some invertible P.

Properties of similar matrices: - Same eigenvalues (with same multiplicities) - Same trace and determinant - Same characteristic polynomial

---

Diagonalization

Definition

Matrix A is **diagonalizable** if it's similar to a diagonal matrix:

$$A = PDP^{-1}$$

where D is diagonal and P is invertible.

Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable iff: 1. A has n linearly independent eigenvectors, OR 2. For each eigenvalue, geometric multiplicity = algebraic multiplicity

Diagonalization Process

1. Find all eigenvalues of A
2. For each eigenvalue, find a basis for its eigenspace
3. If total number of basis vectors equals n, then A is diagonalizable
4. Form P using eigenvectors as columns
5. Form D with corresponding eigenvalues on diagonal

Example

Diagonalize $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

Step 1: Find eigenvalues

$$\det(A - \lambda I) = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 3$

Step 2: Find eigenvectors

For $\lambda_1 = 2$: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 3$: $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Step 3: Form matrices

$$P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Verification: $A = PDP^{-1}$

Applications

Powers of Matrices

If $A = PDP^{-1}$, then:

$$A^k = PD^kP^{-1} = P \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix} P^{-1}$$

Systems of Differential Equations

The system $\mathbf{x}' = A\mathbf{x}$ has solution:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

If $A = PDP^{-1}$, then:

$$e^{At} = Pe^{Dt}P^{-1} = P \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} P^{-1}$$

Principal Component Analysis (PCA)

In data analysis, eigenvalues and eigenvectors of covariance matrices identify: - **Eigenvectors**: Principal directions of data variation - **Eigenvalues**: Amount of variance in each direction

Stability Analysis

For dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$: - **Stable**: All $|\lambda_i| < 1$ - **Unstable**: Any $|\lambda_i| > 1$ - **Marginally stable**: All $|\lambda_i| \leq 1$ with some $|\lambda_i| = 1$

Markov Chains

For stochastic matrix P: - Eigenvalue $\lambda = 1$ always exists - Corresponding eigenvector gives steady-state distribution - Other eigenvalues determine convergence rate

---

Special Types of Matrices

Symmetric Matrices

For symmetric matrix $A = A^T$: - All eigenvalues are real - Eigenvectors corresponding to different eigenvalues are orthogonal - Always diagonalizable: $A = QDQ^T$ where Q is orthogonal

Orthogonal Matrices

For orthogonal matrix Q (where $Q^TQ = I$): - All eigenvalues have absolute value 1 - Over the complex numbers, eigenvalues are of the form $e^{i\theta}$ (points on the unit circle). - Over the reals, eigenvalues are either $+1$ or -1 when a real eigenvalue exists; 2×2 rotation blocks correspond to complex-conjugate pairs $e^{\pm i\theta}$.

Positive Definite Matrices

Matrix A is **positive definite** if: - A is symmetric - All eigenvalues are positive - $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$

Practice Problems

Problem 1

Find eigenvalues and eigenvectors of:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Problem 2

Determine if the following matrix is diagonalizable:

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 3

If A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 3$, find: a) $\text{tr}(A)$ b) $\det(A)$ c) Eigenvalues of A^2 d) Eigenvalues of A^{-1} (if it exists)

Problem 4

For the matrix $C = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$: a) Find eigenvalues and eigenvectors b) Diagonalize C c) Compute C^{10}

Problem 5

Show that if A is invertible, then λ is an eigenvalue of A iff $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Next: Inner Product Spaces and Orthogonality

Solutions

Solution to Problem 1

For $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, the characteristic polynomial is $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$, so $\lambda_1 = 4,\ \lambda_2 = -1$.

Eigenvectors: - For $\lambda = 4$: $(A - 4I)v = 0 \Rightarrow \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} v = 0$, take $v_1 = (2,3)^T$. - For $\lambda = -1$: $(A + I)v = 0 \Rightarrow \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} v = 0$, take $v_2 = (1,-1)^T$.

Solution to Problem 2

Diagonalizable iff there are n linearly independent eigenvectors. Equivalently, for each eigenvalue, geometric multiplicity equals algebraic multiplicity; in total the sum of geometric multiplicities equals n.

Solution to Problem 3

If A is invertible and $Av = \lambda v$ with $v \neq 0$, then $v = A^{-1}(\lambda v) = \lambda A^{-1} v$, so $A^{-1} v = \frac{1}{\lambda} v$. Conversely, if $A^{-1} v = \mu v$, multiply by A to get $v = \mu A v$, hence $Av = \frac{1}{\mu} v$.

Solution to Problem 4

For $C = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$, characteristic polynomial: $\lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6)$. Eigenvectors: - $\lambda = 6$: $(C - 6I)v = 0 \Rightarrow \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} v = 0$, take $v_1 = (2,-1)^T$. - $\lambda = 1$: $(C - I)v = 0 \Rightarrow \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} v = 0$, take $v_2 = (1,2)^T$. Diagonalization: $P = [v_1\ v_2]$, $D = \mathrm{diag}(6,1)$, and $C = PDP^{-1}$.

Solution to Problem 5

If A is invertible, then $Av = \lambda v \Leftrightarrow A^{-1} v = \frac{1}{\lambda} v$. Thus eigenvalues of A^{-1} are reciprocals of nonzero eigenvalues of A, with the same eigenvectors.