Inner Product Spaces and Orthogonality

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Inner Product Spaces

An **inner product** on a real vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying:

Inner Product Axioms

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $c \in \mathbb{R}$:

1. **Symmetry**: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. **Linearity in first argument**:
   - $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
   - $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$
3. **Positive definiteness**: $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality iff $\mathbf{v} = \mathbf{0}$

Standard Inner Product in $\mathbb{R}^n$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

Induced Norm

The inner product induces a **norm**:

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Cauchy-Schwarz Inequality

For any vectors $\mathbf{u}, \mathbf{v}$ in an inner product space:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

Equality holds iff $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

Triangle Inequality

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Orthogonality and Orthonormal Sets

Orthogonal Vectors

Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

Notation: $\mathbf{u} \perp \mathbf{v}$

Orthogonal Sets

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is **orthogonal** if:

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \text{ for all } i \neq j$$

Orthonormal Sets

A set is **orthonormal** if it's orthogonal and each vector has unit length:

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Properties of Orthogonal Sets

1. **Linear Independence**: Any orthogonal set of non-zero vectors is linearly independent
2. **Pythagorean Theorem**: If $\mathbf{u} \perp \mathbf{v}$, then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$
3. **Orthogonal Basis**: An orthogonal basis simplifies many computations

Coordinates in Orthonormal Basis

If $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is an orthonormal basis for V, then:

$$\mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{q}_i \rangle \mathbf{q}_i$$

Gram-Schmidt Process

The **Gram-Schmidt process** converts a linearly independent set into an orthogonal set.

Algorithm

Given linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$:

Step 1: $\mathbf{u}_1 = \mathbf{v}_1$

Step 2: $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1$

Step 3: $\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2$

General Step:

$$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} \frac{\langle \mathbf{v}_k, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \mathbf{u}_j$$

Normalization

To get orthonormal vectors:

$$\mathbf{q}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}$$

Example

Orthogonalize $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Step 1: $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Step 2:

$$\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

Step 3:

$$\mathbf{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1/3}{2/3}\begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

Orthogonal Projections

Projection onto a Line

The **orthogonal projection** of $\mathbf{v}$ onto line spanned by $\mathbf{u}$ is:

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$$

Projection onto a Subspace

For subspace W with orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$:

$$\text{proj}_W\mathbf{v} = \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{q}_i \rangle \mathbf{q}_i$$

Projection Matrix

If $Q = [\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_k]$ has orthonormal columns:

$$P = QQ^T$$

Then $P\mathbf{v} = \text{proj}_W\mathbf{v}$ for any vector $\mathbf{v}$.

Properties of Projection Matrices

1. **Idempotent**: $P^2 = P$
2. **Symmetric**: $P^T = P$
3. **Eigenvalues**: Only 0 and 1

Orthogonal Complement

The **orthogonal complement** of subspace W is:

$$W^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}$$

Fundamental theorem: $V = W \oplus W^\perp$ (direct sum)

QR Decomposition

Every matrix $A \in \mathbb{R}^{m \times n}$ with linearly independent columns can be factored as:

$$A = QR$$

where: - $Q \in \mathbb{R}^{m \times n}$ has orthonormal columns - $R \in \mathbb{R}^{n \times n}$ is upper triangular with positive diagonal entries

Construction via Gram-Schmidt

If $A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n]$, apply Gram-Schmidt to get $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$.

Then:

$$Q = [\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_n]$$

$$R = \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{q}_1 \rangle & \langle \mathbf{a}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{a}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{a}_n, \mathbf{q}_n \rangle \end{pmatrix}$$

Applications of QR Decomposition

1. **Solving linear systems**: $A\mathbf{x} = \mathbf{b} \Rightarrow R\mathbf{x} = Q^T\mathbf{b}$
2. **Least squares problems**
3. **Computing eigenvalues** (QR algorithm)
4. **Numerical stability** in computations

Least Squares Solutions

For overdetermined system $A\mathbf{x} = \mathbf{b}$ (more equations than unknowns), find $\mathbf{x}$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|^2$.

Normal Equations

The least squares solution satisfies:

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

Solution: $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$ (if $A^T A$ is invertible)

Geometric Interpretation

The least squares solution makes $A\mathbf{x}$ the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$:

$$A\mathbf{x} = \mathrm{proj}_{\mathrm{Col}(A)}\mathbf{b}$$

QR Method for Least Squares

If $A = QR$, then:

$$\mathbf{x} = R^{-1} Q^T \mathbf{b}$$

This is more numerically stable than using normal equations.

Applications

1. **Linear regression**: Fitting lines/curves to data
2. **Data fitting**: Polynomial approximation
3. **Signal processing**: Noise reduction
4. **Computer graphics**: Surface fitting

Orthogonal Matrices

A square matrix Q is **orthogonal** if $Q^T Q = I$, equivalently $Q^{-1} = Q^T$.

Properties

1. **Preserves lengths**: $\|Q\mathbf{x}\| = \|\mathbf{x}\|$
2. **Preserves angles**: $\langle Q\mathbf{u}, Q\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$
3. **Determinant**: $\det(Q) = \pm 1$
4. **Eigenvalues**: All have absolute value 1

Examples

- **Rotation matrices**: $\det(Q) = 1$
- **Reflection matrices**: $\det(Q) = -1$
- **Householder reflectors**: $H = I - 2\mathbf{u}\mathbf{u}^T$ where $\|\mathbf{u}\| = 1$

Practice Problems

Problem 1

Apply the Gram-Schmidt process to orthogonalize:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Problem 2

Find the QR decomposition of:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Problem 3

Find the orthogonal projection of $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the subspace spanned by:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Problem 4

Solve the least squares problem for the overdetermined system:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

Problem 5

Show that if Q is orthogonal, then $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for any vector $\mathbf{x}$.

This completes the core linear algebra sequence. Next topics might include: Singular Value Decomposition, Spectral Theory, or applications to specific fields.

Solutions

Solution to Problem 1 (Gram–Schmidt)

Compute

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}.$$

Orthonormal set: $\mathbf{q}_1 = \frac{1}{\sqrt{2}}(1, 0, 1)^T$, $\mathbf{q}_2 = (0, 1, 0)^T$, $\mathbf{q}_3 = \frac{1}{\sqrt{2}}(1, 0, -1)^T$.

Solution to Problem 2 (QR of A)

Apply Gram–Schmidt to columns of A to form Q with orthonormal columns and set $R = Q^T A$. Concretely, the first column normalizes $(1, 1, 0)^T$; the second subtracts its projection and normalizes; then compute R entries via inner products $r_{ij} = \langle a_j, q_i \rangle$.

Solution to Problem 3 (Projection)

With $\mathbf{b} = (1, 2, 3)^T$, $\mathbf{v}_1 = (1, 1, 0)^T$, $\mathbf{v}_2 = (1, 0, 1)^T$, orthonormalize to $Q = [\mathbf{q}_1\ \mathbf{q}_2]$ as in Solution 1, then $\mathrm{proj}_{\mathrm{span}\{v_1, v_2\}}(\mathbf{b}) = QQ^T\mathbf{b}$.

Solution to Problem 4 (Least Squares via QR)

For $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{b} = (2, 3, 5)^T$, QR gives $A = QR$. Solve $R\mathbf{x} = Q^T\mathbf{b}$ to obtain $\mathbf{x} = (1, 1)^T$.

Solution to Problem 5 (Orthogonal preserves norm)

Since $Q^T Q = I$, we have $\|Q\mathbf{x}\|^2 = (Q\mathbf{x})^T (Q\mathbf{x}) = \mathbf{x}^T (Q^T Q)\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$.