Dominance and Iterated Elimination

Game Theory · Module 1 — Strategic Form Games

Node 1.3

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Strategic Form Games **Node:** 1.3

Dominance is the first solution technique in the course — a way to narrow down rational play without computing an equilibrium. A strategy that is worse than another strategy *regardless of what the opponents do* can be deleted from consideration. Iterating this deletion — on the grounds that rational players know rational players delete such strategies — gives the concept of **iterated elimination of dominated strategies**. It is the weakest-possible form of equilibrium reasoning, but surprisingly powerful: some games collapse to a unique profile under iterated strict dominance alone. It also sets up rationalizability (Node 2.5) and provides the sharpest intuition for Nash equilibrium (Module 2).

Formal Definition

Let $G = (N, \{S_i\}, \{u_i\})$ be a finite game, and allow mixed strategies $\Delta(S_i)$ as in Node 1.2.

Strict dominance. A strategy $\sigma_i \in \Delta(S_i)$ **strictly dominates** $s_i \in S_i$ if

$$U_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}.$$

The pure strategy s_i is then said to be **strictly dominated**. Note that the dominating strategy may itself be mixed, and that the inequality is required to hold for every opponent profile.

Weak dominance. σ_i **weakly dominates** s_i if

$$U_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all } s_{-i}, \text{ with strict inequality for at least one } s_{-i}.$$

Iterated elimination of strictly dominated strategies (IESDS). Define the sequence

$$S_i^{(0)} = S_i, \qquad S_i^{(k+1)} = \{s_i \in S_i^{(k)} : s_i \text{ is not strictly dominated given } S_{-i}^{(k)}\},$$

where the dominance check in the $(k+1)$-th round uses mixtures over $S_i^{(k)}$ and opponent profiles in $S_{-i}^{(k)}$. The set of strategies surviving iterated elimination is

$$S_i^{\infty} = \bigcap_{k \geq 0} S_i^{(k)}.$$

A game is **dominance-solvable** if $|S_i^{\infty}| = 1$ for every i.

Intuition

A strictly dominated strategy is never optimal. Not once. Not against any opponent. A rational player never plays it. So, the reasoning goes, a rational player's choice must lie in $S_i \setminus \{\text{strictly dominated}\}$. A rational player who knows the opponent is rational also knows the opponent's choice lies in the opponent's reduced set, and can therefore delete any of their own strategies that are strictly dominated *on the reduced opponent set*. This recursive structure — "I am rational, I know you are rational, I know you know I am rational" — is **common knowledge of rationality**, and it is exactly what iterated elimination operationalizes.

Two warnings:

Strict vs weak. Iterated elimination of *strictly* dominated strategies is order-independent: whichever order you delete in, you get the same S_i^∞. Iterated elimination of *weakly* dominated strategies is not order-independent — different orders give different surviving sets. This makes weak dominance a delicate solution concept and strict dominance a clean one.

Mixing matters. A pure strategy can be strictly dominated by a mixture without being strictly dominated by any single pure strategy. So iterated elimination *must* allow mixtures on the dominating side, or it will under-delete.

Structural Role

Dominance is upstream of every equilibrium concept. Specifically:

- **Any Nash equilibrium lies in S^∞.** A strictly dominated strategy cannot be a best response, so cannot be in the support of any equilibrium.
- **Rationalizability (2.5)** is the fixed point of a similar elimination procedure that uses best-response correspondences instead of dominance. It is a strictly weaker concept than dominance-solvability for 3+ players and coincides with iterated strict dominance for 2 players.
- **Dominance-solvable games** are exactly the games where common knowledge of rationality pins down a unique profile — no further equilibrium assumption is needed.

Dominance also acts as a *pre-processor*: before computing Nash equilibria of a big game, you can delete all strictly dominated strategies and compute on the reduced game, and every Nash equilibrium of the original is an equilibrium of the reduced game, and vice versa.

Mathematical Development

Order-independence for strict dominance

Theorem. The limit S_i^∞ of iterated strict-dominance elimination is independent of the order in which strategies are deleted.

Sketch. Strict dominance is a monotone operator: deleting a strategy never revives a previously dominated one, because the inequality $U_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ continues to hold when opponent profiles are restricted to a subset. The intersection of all maximal sequences of deletions is therefore the same set.

Non-monotonicity for weak dominance

Consider

$$\begin{array}{c|cc} & L & R \\ \hline T & (1,1) & (0,0) \\ M & (1,1) & (2,1) \\ B & (0,0) & (2,1) \end{array}.$$

T is weakly dominated by M; B is weakly dominated by M. Different orders of elimination lead to different survivors, because weak dominance is lost when opponent strategies supporting the strict inequality are themselves deleted first.

Mixed dominance exceeds pure dominance

	L	R
T	$(3,0)$	$(0,0)$
M	$(0,0)$	$(3,0)$
B	$(1,0)$	$(1,0)$

B is not dominated by T alone (fails against L) or by M alone (fails against R). But the mixture $\frac{1}{2}T + \frac{1}{2}M$ yields 1.5 in both columns, strictly better than B's 1. So B is strictly dominated by a mixture.

Dominance-solvable games

Cournot duopoly with linear demand is dominance-solvable: iteratively eliminating never-best-response quantities converges to the unique Cournot equilibrium. **Iterated beauty contest** (guess a number, the winner is closest to p times the average, with $p < 1$) is dominance-solvable: successive rounds of "no one rational picks above $100p$", "no one rational picks above $100p^2$", and so on, collapse the game to 0. These are the canonical demonstrations that iterated strict dominance has real solving power.

Worked Example

A 3×3 elimination

	L	C	R
T	$(4,3)$	$(1,2)$	$(2,1)$
M	$(3,2)$	$(2,1)$	$(1,3)$
B	$(0,0)$	$(0,0)$	$(1,1)$

Round 1 (Column). Column's payoffs are $L : (3,2,0)$, $C : (2,1,0)$, $R : (1,3,1)$. Try the mix $(0.6L, 0.4R)$: $(0.6 \cdot 3 + 0.4 \cdot 1, 0.6 \cdot 2 + 0.4 \cdot 3, 0.6 \cdot 0 + 0.4 \cdot 1) = (2.2, 2.4, 0.4)$. Compared to $C = (2,1,0)$, the mixture strictly dominates C. Delete C.

Round 2 (Row, on reduced game).

	L	R
T	$(4,3)$	$(2,1)$
M	$(3,2)$	$(1,3)$
B	$(0,0)$	$(1,1)$

.

Row payoffs: $T : (4,2)$, $M : (3,1)$, $B : (0,1)$. T strictly dominates M $(4 > 3, 2 > 1)$. Delete M. Does T strictly dominate B? $4 > 0$ and $2 > 1$ — yes. Delete B.

Round 3. Row is pinned to T. Column chooses between $(4,3)$ and $(2,1)$; Column prefers L. The game is dominance-solvable with outcome (T, L), payoff $(4,3)$.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Strictly dominated strategy	A line that is EV-worse than another line for *every* opponent response (open-limping 72*o* UTG is dominated by folding)
Mixed dominance	A mixture of two lines beats a pure line — e.g., balanced bet/check strictly dominates "always min-raise" with a polarized holding
IESDS	Preflop range construction: "don't open into 4 limpers with $K6o$" is iterated dominance
Dominance-solvable subgame	Face-up river decisions in a solver — the best action is often pinable by pointwise comparison
Rationalizability preview	The set of "defensible plays" — the GTO range of openable hands

Concrete situation

UTG in a 9-handed \$1/\$2 live game, stack 100 BB. You are dealt 7♠2♡.

Pure-strategy analysis. Your actions are Fold, Limp, Open-raise. Against any reasonable opponent strategy profile, Fold strictly dominates Limp: your EV from limping 72*o* UTG is negative (you will rarely flop anything playable, and when you do the effective stack is deep enough that you lose more than you win). Fold also strictly dominates Open-raise: raising 72*o* from UTG is burning chips. Hence both Limp and Open-raise are strictly dominated by Fold, and an IESDS pre-processing eliminates them.

Repeat this for every garbage hand in UTG. After one round of IESDS you have the UTG opening range: roughly the top 15% of hands open, everything else folds.

Now move to MP. MP's decision depends on what UTG did. If UTG opened, MP knows UTG has a strong range (top 15%), so MP's strategy is computed in the reduced game. Some MP hands that would have been opens in isolation (say, ATo) become dominated by fold *in the reduced game* because they fare badly against UTG's concentrated strong range. This is IESDS iterating: round 2 eliminates strategies that were only rational under round-0 opponent behavior.

What this understanding buys you

The preflop ranges you memorize are the output of IESDS. The charts you study are not arbitrary — they are the dominance-reduced action set. When you see a hand and cannot recall the chart, you can often reconstruct it by asking: "Is there any opponent response under which opening this hand is better than folding?" If the answer is no under every reasonable opponent range (the set that *they* have not had eliminated), then folding is the rational play.

Second lesson: the hands you agonize over are exactly the ones *not* eliminated by dominance — the rationalizable but non-dominance-pinned hands. The AJo from MP, the suited connectors UTG+1. Those are the hands where equilibrium mixing matters. For all the obviously foldable and obviously raiseable hands, dominance already tells you what to do, and you should spend no thought at all.

Cross-Links

- **1.1 Players, strategies, payoffs** — the objects dominance operates on.
- **1.2 Mixed strategies** — required for mixed dominance.
- **1.4 Best response** — the functional companion to dominance.
- **2.1 Nash equilibrium** — every Nash profile survives IESDS.
- **2.5 Rationalizability** — the best-response analogue of iterated dominance.
- **Economics — Cournot and Bertrand** — classic dominance-solvable models.

- **AI Systems — minimax pruning** — dominance as tree-pruning in game search.

Structural Compression

Invariant: A strictly dominated strategy can never be a best response, for any belief about opponents; consequently no rational player ever plays one, and no equilibrium assigns it positive weight.

Mechanism: Pointwise comparison of payoff functions across opponent profiles, extended to mixtures to capture all dominance relations, iterated over rounds that model increasing depth of common knowledge of rationality.

Cross-System Echo: Dominance plays the role of "obviously irrational" pruning in many settings: dead-code elimination in compilers, constraint propagation in CSPs, alpha-beta pruning in game tree search, strictly-dominated action elimination in multi-armed bandits. In each case, a monotone pre-processing step removes options that can be proved suboptimal without a full solution.

Exercises

1. Apply IESDS to the Prisoner's Dilemma. Show that Defect strictly dominates Cooperate for each player, and conclude the game is dominance-solvable with outcome (D, D).
2. In the Stag Hunt of Node 1.1, show that *no* strategy is strictly dominated. (Dominance gives no information — equilibrium refinement is needed.)
3. Construct a 2×3 game in which a pure strategy of Row is strictly dominated by a mixture of two other pure strategies but by no single pure strategy.
4. Prove order-independence of IESDS: if s_i is strictly dominated at some round of elimination, and s_i' is eliminated in that round, then s_i remains strictly dominated in the reduced game.
5. In the p-beauty contest with $p = 2/3$ on the interval $[0, 100]$, show that after k rounds of IESDS, every surviving strategy is in $[0, 100(2/3)^k]$. Conclude the game is dominance-solvable with unique outcome 0.
6. Show that if a strategy profile s^* is a (pure) Nash equilibrium, then each $s_i^* \in S_i^\infty$. (Contrapositive: a dominated strategy cannot be part of any equilibrium.)
7. **Argue, structurally,** why iterated elimination of *weakly* dominated strategies is order-dependent while iterated elimination of *strictly* dominated strategies is not. Identify the precise step in the order-independence argument that uses strictness of the inequality.

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Concepts: dominance
Prerequisites: 1.1, 1.2
Enables: 1.4, 2.1, 2.5

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