Normal Form as Universal Representation

Game Theory · Module 1 — Strategic Form Games

Node 1.6

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Strategic Form Games **Node:** 1.6

The normal form (N, S, u) from Node 1.1 is the minimal specification of a strategic situation. This node makes the stronger claim: *every* game, including sequential and information-rich ones introduced in Modules 3 and 4, can be flattened back to normal form — at the cost of an exponentially larger strategy space. This universality is what lets Module 2's equilibrium theory apply to the whole subject; it is also a warning, because the flattened form loses information that is visible in extensive form. The node is the formal interface between strategic form (Module 1) and extensive form (Module 3), and it motivates the equilibrium refinements of subgame perfection (3.5) and sequential equilibrium (4.6).

Formal Definition

Let Γ be an extensive-form game with player set N, a rooted game tree T, information partitions $\{\mathcal{I}_i\}_{i\in N}$, action assignments, chance moves with known probabilities, and outcome payoffs. (Module 3 develops this structure.)

A **behavioral strategy** for player i is a map $\beta_i : \mathcal{I}_i \to \Delta(A)$, assigning to each of i's information sets a distribution over the actions available at that set.

A **pure strategy** s_i in extensive form is a map $s_i : \mathcal{I}_i \to A$, assigning to each information set a single action. The set of pure strategies of player i is

$$S_i^{\text{ext}} = \prod_{I\in\mathcal{I}_i} A(I).$$

The **normal form of** Γ is the strategic form game $G(\Gamma) = (N, \{S_i^{\text{ext}}\}, \{u_i^{\text{ext}}\})$, where the payoff $u_i^{\text{ext}}(s_1, \dots, s_n)$ is the expected payoff to player i along the stochastic path induced by $(s_1, \dots, s_n)$ through the tree T, averaged over chance moves.

Universal representation claim. Every finite extensive-form game Γ has a well-defined normal form $G(\Gamma)$, and every Nash equilibrium of $G(\Gamma)$ corresponds to a strategy profile of Γ. Conversely, every Nash equilibrium of Γ (interpreted as a behavioral profile) induces a Nash equilibrium of $G(\Gamma)$ via Kuhn's theorem on the equivalence of mixed and behavioral strategies under perfect recall.

Combinatorial cost. If player i has k_I actions at information set I, then $|S_i^{\text{ext}}| = \prod_{I\in\mathcal{I}_i} k_I$, which is typically exponential in the number of information sets.

Intuition

A pure strategy in the normal form is a **complete contingent plan** — a specification of what you would do at every decision point you could reach. In a game with a single decision, the plan is just your action. In a game with many decisions and branches, the plan must cover every branch, including ones you would never actually visit under rational play.

Flattening an extensive-form game to normal form means enumerating all such plans. For chess (if it were finite and deterministic), a pure strategy would be a function specifying your move in every possible board position. The strategy space is astronomical.

But the normal form is still *uniquely* defined, and the abstract theory of Module 2 — existence, mixed equilibria, rationalizability — applies to it without modification. What's lost is the *temporal structure*: the normal form treats your entire contingent plan as a single choice, ignoring the fact that in the real game you'd choose step-by-step and update your plan as you learn new information. Module 3 will recover this structure via subgame perfection, and Module 4 via sequential equilibrium.

A key principle: **Kuhn's theorem** (1953) says that under *perfect recall* (a player never forgets their own past actions or observations), mixed strategies in the normal form are strategically equivalent to behavioral strategies in the extensive form. Both representations yield the same distribution over terminal nodes and the same expected payoffs.

Structural Role

This node closes Module 1 and opens the door to Modules 3 and 4:

- **1.1–1.5 ⇒ 1.6.** The entire machinery of strategic-form analysis (dominance, best response, minimax, mixed equilibria) applies to any extensive-form game via its normal form.
- **1.6 ⇒ 2.4.** Nash existence in extensive form is an immediate corollary of Nash existence in normal form plus this universality.
- **1.6 ⇒ 3.3.** Extensive-form strategies are formally defined in Module 3; this node previews the enumeration.
- **1.6 ⇒ 3.5.** Subgame perfection is a refinement that excludes normal-form Nash equilibria that rely on non-credible off-path behavior.
- **1.6 ⇒ 4.6.** Sequential equilibrium refines further using consistent beliefs at out-of-equilibrium information sets.
- **Practical corollary.** "Solving" a sequential game can be done either on the normal form (LP for zero-sum, fixed point for general-sum) or directly on the extensive form (backward induction, CFR, tree search). The existence theorems are about the normal form; the efficient algorithms are about the extensive form.

Mathematical Development

From tree to table

Given extensive-form game Γ, enumerate all pure strategies for each player and compute payoffs for each strategy profile. This is the bijective construction of $G(\Gamma)$.

Example. Consider the game: player 1 moves first, choosing L or R. If L, payoff $(2, 0)$. If R, player 2 chooses l or r. If r, payoff $(0, 1)$; if l, payoff $(1, 2)$.

- Player 1 has 2 pure strategies: $\{L, R\}$.
- Player 2 has 2 pure strategies: $\{l, r\}$ — even though player 2 only moves in the R subtree, the pure strategy is defined for all information sets; here there is only one information set for player 2.
- Payoff matrix (rows = player 1, columns = player 2):

	l	r
L	$(2,0)$	$(2,0)$
R	$(1,2)$	$(0,1)$

Notice player 1's L row is constant in player 2's action: L "closes" the game before player 2 moves. This is an artifact of the flattening: in the tree, player 2 doesn't act at all after L.

Exponential blowup

For a depth-d binary tree where each player moves on alternating levels, each player has $2^{d/2}$ information sets and $2^{2^{d/2}}$ pure strategies. For $d = 20$, that is already 2^{1024} — vastly more than the number of atoms in the observable universe.

Kuhn's theorem on behavioral strategies

Theorem (Kuhn, 1953). In a finite extensive-form game with perfect recall, for every mixed strategy $\sigma_i \in \Delta(S_i^{\text{ext}})$ there exists a behavioral strategy β_i such that σ_i and β_i induce the same distribution over terminal nodes, and conversely.

The significance: you never need to reason about $\Delta(S_i^{\text{ext}})$, which is doubly exponential; you can instead work with behavioral strategies β_i, which live in the product of simplices $\prod_{I \in \mathcal{I}_i} \Delta(A(I))$, which is only linear in the number of information sets. Without Kuhn's theorem, computational game theory would be infeasible.

Caveat. Kuhn's theorem fails under **imperfect recall** — when a player forgets their own past actions or observations. In such games, mixed and behavioral strategies are not equivalent. This matters in abstraction-based poker solving, where information-set merging introduces imperfect-recall artifacts.

Payoff-equivalence of representations

Two games G and G' are **strategically equivalent** if they have the same set of Nash equilibria (identified via any bijection on strategies). The normal form of an extensive-form game is strategically equivalent to the extensive form in this sense, but it is not *refinement-equivalent*: some Nash equilibria of the normal form fail subgame perfection in the extensive form. Module 3 will develop this.

Worked Example

Subgame-perfect failure in the normal form

Consider the entry deterrence game: an Entrant (player 1) decides whether to enter a market (In) or stay out (Out). If In, the Incumbent (player 2) decides to Fight (F) or Accommodate (A).

Extensive form payoffs: Out $\to (0, 2)$; In, A $\to (1, 1)$; In, F $\to (-1, -1)$.

Normal form:

	F	A
In	$(-1,-1)$	$(1,1)$
Out	$(0,2)$	$(0,2)$

Nash equilibria of the normal form. Two pure ones: 1. (In, A): payoff $(1, 1)$. Both best-respond. 2. (Out, F): payoff $(0, 2)$. Entrant best-responds given Incumbent threatens to Fight; Incumbent best-responds given Entrant is Out (Incumbent never has to actually fight, so any action is a best response on the unreached subgame).

What's lost in the flattening. In the extensive form, equilibrium #2 relies on an *incredible* threat: if the Entrant actually entered, the Incumbent's payoff from Fight is -1 vs Accommodate's $+1$. The threat to fight is not credible — it would be suboptimal to execute. In the normal form, this incredibility is invisible because "F" on the off-path branch never lowers Incumbent's realized payoff. This is the motivating example for subgame perfection (Node 3.5).

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Extensive form Γ	The full betting tree of a hand, including decision points, action branches, and chance (dealt cards)
Information set	A set of game histories indistinguishable to the acting player (unknown hole cards, unknown previous mixed-action resolutions)
Pure strategy in extensive form	A complete contingent plan across every hole-card/board/history combination
Normal-form enumeration	Astronomical — HUNL has ~10^{161} information sets, making full normal form infeasible
Behavioral strategy	The object a solver actually stores — frequency distributions over actions at each information set
Kuhn's theorem	Why CFR can output behavioral frequencies and still be "solving" the mixed-strategy game

Concrete situation

HUNL flop-and-beyond decision tree. After the flop, each player can check, bet (multiple sizes), call, fold, or raise (multiple sizes). A reasonable abstraction uses 3 bet sizes and 2 raise sizes, producing ~7 actions per decision. A typical flop–turn–river tree has 20–40 decision points per hand, and with 1326 hole-card combinations and 22100 boards, the number of information sets is in the hundreds of millions even after aggressive abstraction. The pure-strategy count is 7^{millions} — a number whose *description* is longer than this lesson.

Nobody enumerates the normal form. Instead, modern solvers (PioSolver, MonkerSolver, GTO+) store **behavioral strategies**: frequency distributions at each information set. Kuhn's theorem guarantees this is strategically equivalent to solving the normal form. CFR iterates on behavioral strategies directly, updating frequencies by counterfactual regret, and converges to a near-Nash equilibrium of the normal form *as measured by exploitability.*

What this understanding buys you

You never have to think about the normal form of a poker game. The strategy you are studying, the frequencies the solver outputs, the "GTO mix" you are trying to memorize — these are behavioral strategies in the extensive-form tree. Kuhn's theorem guarantees they are strategically equivalent to mixed strategies in the normal form, which is the object the existence theorems apply to. You get the abstract theory's guarantees (Nash equilibrium exists, zero-sum games have a value, etc.) without ever having to enumerate 7^{millions} contingent plans.

The warning is about abstraction. When a solver abstracts the game (e.g., by bucketing similar hands together), the abstraction may introduce imperfect recall, and Kuhn's theorem may fail. This means the solution on the abstracted tree is not guaranteed to be a Nash equilibrium of the original game. Libratus and DeepStack handled this with careful subgame re-solving and abstraction-refinement techniques to bound the error.

Cross-Links

- **1.1 Players, strategies, payoffs** — the base objects.
- **1.3 Dominance** — dominated strategies in the normal form of an extensive-form game.
- **3.1 Game trees** — the extensive form itself.
- **3.3 Strategies in extensive form** — the formal definition of s_i as a function on information sets.
- **3.5 Subgame perfect equilibrium** — the refinement that distinguishes normal-form Nash equilibria on the basis of off-path credibility.
- **4.6 Sequential equilibrium** — further refinement with consistent beliefs.
- **9.3 Counterfactual regret minimization** — the algorithm that exploits behavioral-strategy representation.
- **Statistics — conditional expectation** — how payoffs are computed over randomness.

Structural Compression

Invariant: Every finite extensive-form game has a unique normal-form representation obtained by enumerating complete contingent plans; under perfect recall, mixed and behavioral strategies yield equivalent solution concepts.

Mechanism: Contingent-plan enumeration plus payoff computation along all induced paths through the tree; Kuhn's theorem converts the doubly-exponential mixed-strategy space to the linear-in-information-sets behavioral-strategy space.

Cross-System Echo: The reduction of sequential/stateful problems to flat, stateless representations is a recurring move in computer science: POMDPs reduce to belief MDPs, Markov games reduce to matrix games at each state, policy spaces in reinforcement learning are the behavioral-strategy analogue of the normal-form flattening. The computational-versus-representational tradeoff is universal.

Exercises

1. Write out the full normal form of the entry deterrence game above. Identify both pure Nash equilibria and verify which is subgame perfect (preview of 3.5).

2. Consider a two-stage game: player 1 chooses $\{L, R\}$; player 2 observes player 1's choice and then chooses $\{l, r\}$. Payoffs: $LL \to (1,1), LR \to (0,2), RL \to (2,0), RR \to (-1,-1)$. Write the normal-form payoff matrix. Player 2 has how many pure strategies? Find all Nash equilibria.

3. For a perfect-information game of depth d with branching factor 2, count the pure strategies of the first player assuming they move at every other depth.

4. State Kuhn's theorem precisely and sketch why it requires perfect recall.

5. Give an example of a game with imperfect recall in which a mixed strategy in the normal form achieves a payoff unattainable by any behavioral strategy.

6. Verify that if a normal-form equilibrium assigns positive probability to a strategy that prescribes a strictly dominated action at an off-path information set, the equilibrium fails subgame perfection.

7. **Argue, structurally,** why normal-form equivalence is necessary but not sufficient for a satisfying solution concept in extensive-form games. Identify one concrete piece of strategic information that the normal form preserves and one it loses.

Game Theory · Module 1 — Strategic Form Games · Node 1.6
Concepts: graph-theory, matrix-representation

Prerequisites: 1.1, 1.2, 1.3, 1.4, 1.5
Enables: 2.1, 3.1, 3.3

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