Pure-Strategy Nash Equilibrium

Game Theory · Module 2 — Nash Equilibrium

Node 2.2

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Nash Equilibrium **Node:** 2.2

This node specializes Nash equilibrium (2.1) to the case where each player plays a deterministic strategy. Pure Nash equilibria are the easiest to find, the easiest to interpret, and the ones with the strongest stability properties, but they are not guaranteed to exist in every game. Recognizing which games have pure equilibria — and which classes of games are *guaranteed* to have them (supermodular games, potential games, finite games with strategic complementarities) — is one of the main applied-theory skills in game theory. This node also establishes pure-strategy analysis via best-response matrix inspection, the "find all pairs of marked cells" technique that is used throughout the course.

Formal Definition

A **pure-strategy Nash equilibrium** of a game $G = (N, \{S_i\}, \{u_i\})$ is a profile $s^* = (s_1^*, \ldots, s_n^*) \in S = \prod_i S_i$ such that for every player i and every $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Equivalently, $s_i^* \in \mathrm{BR}_i(s_{-i}^*)$ for every i, where best responses are taken over pure strategies only.

Strict pure Nash. $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$ for all $s_i \neq s_i^*$. Every strict Nash equilibrium is pure (you cannot strictly improve on a mixture over ties).

Pure Nash need not exist. Matching Pennies has no pure Nash equilibrium. Rock–Paper–Scissors has no pure Nash. Any game in which the best-response correspondence cycles without a fixed point in pure strategies lacks one.

Games that guarantee pure Nash. Several classes: - **Finite dominance-solvable games** (the surviving profile is a pure Nash). - **Finite two-player games with an ordinal potential function** (Monderer–Shapley). - **Supermodular games on lattices** (Topkis, Milgrom–Roberts): always have smallest and largest pure equilibria. - **Games with strategic complementarities** (continuous-action version of the above). - **Weakly acyclic games**: every better-response path ends at a pure equilibrium.

Intuition

A pure Nash equilibrium is a "frozen" configuration: each player is playing a single strategy, and knows the others are too, and would not want to move. It is the cleanest form of stability.

Two practical diagnostic techniques for finding pure Nash equilibria by hand:

Double-underline method. In each column, underline Row's best payoff. In each row, underline Column's best payoff. A cell with *both* numbers underlined is a pure Nash equilibrium. This works immediately for 2×2 up to 5×5 bimatrix games.

Best-response diagram. Draw Row's best-response set as a function of Column's strategy, and vice versa. Intersections are Nash equilibria. This works for continuous-action games.

The existence of pure equilibria in specific game classes is the content of several classical theorems: Topkis (1978) for supermodular games, Monderer–Shapley (1996) for potential games, Debreu–Fan–Glicksberg for concave games. Each exploits a structural feature of the payoff functions to reduce pure-Nash existence to a fixed-point result.

Structural Role

Pure Nash equilibria are the conceptually clean subset of Nash equilibria:

- **2.1 ⇒ 2.2.** Pure Nash are the special case of Nash where all strategies are deterministic.
- **2.2 ⇒ 2.3.** If a game has no pure Nash, you must look for mixed ones.
- **2.2 ⇒ 3.5.** Subgame perfection often picks out a specific pure Nash in extensive-form games with sequential structure.
- **2.2 ⇒ 5.4.** Nash reversion strategies in repeated games can be "stay at pure Nash forever" — pure is easiest to coordinate on.
- **2.2 ⇒ 7.2.** An evolutionarily stable strategy (ESS) must be a strict Nash, which is necessarily pure.

Pure Nash is also the computationally easiest class: checking if a given profile is pure Nash takes time linear in the game description. Finding one is NP-hard in general, but polynomial for supermodular, potential, and zero-sum games.

Mathematical Development

Potential games

A game G is a **potential game** (Monderer–Shapley, 1996) if there exists a function $\Phi : S \to \mathbb{R}$ such that for every player i, every s_{-i}, and every s_i, s_i',

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}).$$

That is, any player's unilateral change in payoff equals the change in Φ. Every maximizer of Φ over S is a pure Nash equilibrium. Since Φ is a real function on a finite set, it has a maximum, so every finite potential game has a pure Nash.

Examples. Congestion games (Rosenthal, 1973). Coordination games with symmetric additive payoffs. Quadratic network games.

Supermodular games

A game is **supermodular** if the strategy sets are lattices and the payoff functions have **increasing differences** in own strategy and opponent profile: for $s_i > s_i'$ and $s_{-i} > s_{-i}'$,

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}') - u_i(s_i', s_{-i}').$$

This says: the marginal benefit of raising my strategy is higher when my opponents' strategies are higher. Equivalently, best responses are weakly increasing in opponents' strategies (strategic complementarities).

Topkis's theorem. In a supermodular game, the set of pure Nash equilibria is non-empty and has a smallest and largest element with respect to the lattice order.

Best-response dynamics and cycles

Consider Matching Pennies. Row's best response to Column playing Heads is Heads. Column's best response to Row playing Heads is Tails. Row's best response to Column playing Tails is Tails. Column's best response to Row playing Tails is Heads. The best-response dynamics cycles: $HH \to HT \to TT \to TH \to HH$. No pure fixed point exists.

Weakly acyclic games are those where from every profile, there exists a finite sequence of better-response moves ending at a pure Nash. Matching Pennies is not weakly acyclic.

Counting pure equilibria

For a random game with iid uniform payoffs on $[0, 1]$, the expected number of pure Nash equilibria in an $m \times m$ bimatrix game is approximately 1 for large m. This is an interesting fact: "typical" finite games have roughly one pure equilibrium on average, but the variance is large — some have several, many have none.

Worked Example

A 3×3 bimatrix with multiple pure equilibria

	L	C	R
T	$(3, 2)$	$(1, 3)$	$(2, 1)$
M	$(4, 0)$	$(2, 2)$	$(3, 1)$
B	$(1, 4)$	$(0, 1)$	$(5, 3)$

Underline Row's best in each column. Column L: $\underline{4}$ (M). Column C: $\underline{2}$ (M). Column R: $\underline{5}$ (B).

Underline Column's best in each row. Row T: $\underline{3}$ (C). Row M: $\underline{2}$ (C). Row B: $\underline{4}$ (L).

Double-underlined cells. Cell (M, C): Row best in column C (4 underlined... wait, let me recheck. Row payoffs in column C: $1, 2, 0$, best is M with 2 — correct. Column payoffs in row M: $0, 2, 1$, best is C with 2 — correct.) So (M, C) is a pure Nash equilibrium with payoff $(2, 2)$.

Check other double-underlined cells: (B, R)? Row best in R is B with 5 ✓. Column best in B is L with 4; C gives 1, R gives 3. Column's best in row B is L, not R. So R is not best for Column in row B. (B, R) is not a Nash equilibrium.

Unique pure Nash: (M, C) with payoff $(2, 2)$.

A game with no pure Nash

	L	R
T	$(0, 1)$	$(1, 0)$
B	$(1, 0)$	$(0, 1)$

This is Matching Pennies (relabeled). Best responses cycle. No pure Nash. Module 2.3 will find the mixed equilibrium $((1/2, 1/2), (1/2, 1/2))$.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Pure strategy profile	A 100/0 assignment for every hand in every spot — no mixing
Pure Nash equilibrium	A game configuration where every player's pure strategy is a best response to every other's
Non-existence of pure Nash	Most interesting poker spots — bluff frequencies, bluff-catcher frequencies, polarized river bets — have no pure equilibrium
Pure Nash in simple subgames	Face-up river spots where betting is unambiguously best — no mixing required
Strict pure Nash	A "locked" decision: say, folding 72o to a 4-bet — nothing ambiguous, no indifference

Concrete situation

A pocket pair on the button in a heads-up spot where villain is known to be going all-in with their entire range. You hold QQ with 100 BB effective. Villain shoves. Do you call?

Pure-strategy equilibrium subgame. At this exact information set, calling and folding are the only actions. Calling with QQ against a shove-with-any-two-cards range has ~85% equity, a clear +EV call. Folding is strictly worse. The best response is pure: Call. Villain's best response, given the all-in shoving strategy is fixed, is also pure (by construction). So in this specific subgame the pure strategy profile "(Hero calls with QQ, Villain jams any two)" is a pure Nash equilibrium of the restricted game.

In a more realistic equilibrium, Villain would not shove any two cards, and the game would have a mixed component — but the present conditional subgame is pure.

Contrast: bluff-catching. On a river with a polarized betting range from opponent, your J-high bluff-catcher typically has *mixed* best response: call with some frequency, fold with the rest. There is no pure Nash at that information set because pure "always call" is exploitable by more bluffs, and pure "always fold" is exploitable by more value. Indifference forces mixing.

What this understanding buys you

When you are in a pure-Nash spot, don't mix. If the subgame has a dominant strategy (say, calling AA vs a shove), no mixing is required; the equilibrium is deterministic. Mixing in a pure-Nash spot is just random noise.

When you are in a mixed-Nash spot, mix. If the EV of two actions is within 0.1 BB/100 of each other in equilibrium, the spot is a mixing spot, and you need an external randomizer to hit the right frequency.

The practical diagnostic. Before agonizing over a decision, ask "is this a pure-Nash spot or a mixed-Nash spot?" If the answer is pure, stop thinking and take the action. If the answer is mixed, you should have *already* decided the frequency and just execute. No pure-Nash spot should take more than a second; no mixed-Nash spot should take more than a second *either*, because the EV difference is negligible.

Cross-Links

- **1.4 Best-response correspondences** — the underlying operator.
- **2.1 Nash equilibrium definition** — the parent concept.
- **2.3 Mixed Nash** — the case that arises when no pure Nash exists.
- **2.4 Existence** — pure Nash not guaranteed; mixed Nash is.
- **3.5 Subgame perfection** — often selects a pure subgame-perfect Nash.
- **Economics — potential games, supermodular games** — classes guaranteeing pure Nash.

- **Network economics — congestion games** — a classical potential-game family.

Structural Compression

Invariant: A pure Nash equilibrium is a profile s^* of deterministic strategies such that every player's action is a best response to the others' actions.

Mechanism: Pointwise verification of the best-response condition on every player. For structured classes (potential, supermodular), existence follows from maximizing a single real-valued function or from lattice-theoretic fixed-point theorems.

Cross-System Echo: Pure equilibria correspond to "deterministic attractors" in dynamical systems — configurations where no variable has a reason to change. Contrast mixed equilibria, which correspond to "stable mixing distributions" in ergodic dynamical systems.

Exercises

1. Use the double-underline method to find all pure Nash equilibria of the 3×3 game

$$\begin{array}{c|ccc} & L & C & R \\ \hline T & (1,0) & (2,3) & (0,1) \\ M & (3,1) & (1,2) & (4,2) \\ B & (0,3) & (2,0) & (3,3) \end{array}.$$

2. Show that the Prisoner's Dilemma is a potential game by exhibiting a potential function.
3. Prove that every strict Nash equilibrium is pure.
4. Show that a congestion game with cost functions $c_e(x_e)$ on edges has the Rosenthal potential $\Phi(s) = \sum_e \sum_{k=1}^{x_e(s)} c_e(k)$ and hence has a pure Nash equilibrium.
5. For the Stag Hunt, compute both pure Nash equilibria and identify which is Pareto-efficient and which is risk-dominant.
6. Construct a 2×2 game with no pure Nash but with continuous payoffs, demonstrating that the lattice hypothesis of Topkis's theorem is necessary for pure existence in supermodular games.
7. **Argue, structurally,** why the existence of a pure Nash equilibrium is a *fragile* property — specifically, why small perturbations of payoffs can destroy pure equilibria but not mixed ones. What does this suggest about the relative robustness of pure-strategy and mixed-strategy analysis?

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Concepts: nash-equilibrium, dominance
Prerequisites: 1.4, 2.1
Enables: 2.3, 2.4, 3.5

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