Mixed-Strategy Nash Equilibrium

Game Theory · Module 2 — Nash Equilibrium

Node 2.3

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Nash Equilibrium **Node:** 2.3

Mixed Nash is where the deep machinery of game theory lives. Most games of strategic interest have no pure Nash equilibrium, and those that do usually have mixed equilibria as well. Mixed Nash is the object that Nash existence (2.4) provides unconditionally for finite games, and it is the object that modern solvers (CFR, fictitious play, Lemke–Howson) actually compute. This node develops the support characterization, the indifference principle, and the practical technique of guessing supports, writing indifference equations, and solving them — the computational rhythm of mixed-equilibrium analysis.

Formal Definition

A **mixed-strategy Nash equilibrium** of a finite game $G = (N, \{S_i\}, \{u_i\})$ is a profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*) \in \prod_{i \in N} \Delta(S_i)$ such that for every player i and every $\sigma_i \in \Delta(S_i)$,

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*).$$

Support characterization. σ^* is a mixed Nash equilibrium iff for every i, every pure strategy in $\mathrm{supp}(\sigma_i^*)$ yields the same expected payoff against σ_{-i}^*, and this common payoff is at least the payoff of any pure strategy outside the support:

$$U_i(s_i, \sigma_{-i}^*) = v_i^*,\ s_i \in \mathrm{supp}(\sigma_i^*); \qquad U_i(s_i, \sigma_{-i}^*) \leq v_i^*,\ s_i \notin \mathrm{supp}(\sigma_i^*).$$

Here $v_i^* = U_i(\sigma_i^*, \sigma_{-i}^*)$ is the equilibrium payoff of player i.

Completely mixed Nash. An equilibrium in which every $s_i \in S_i$ is in the support of σ_i^*. This requires $|S_i| - 1$ equality constraints per player.

Indifference principle. If any mixed strategy is a best response and its support has more than one element, those elements must all yield the same expected payoff.

Intuition

A mixed equilibrium has a subtle self-referential quality. The player who is mixing is *indifferent* between the pure strategies in their mix — they do not personally prefer one over the other. So why do they mix? Not to optimize their own payoff (any pure strategy in the support is equally good), but to keep the *opponent's* best response correct. If the mixing player deviated to a pure strategy, the opponent would have a new best response, and the whole equilibrium would unravel.

The mathematical content of "mixing to keep the opponent honest" is exactly the support characterization: *my* mix solves an indifference equation for *the opponent*, and vice versa.

This makes mixed equilibria counterintuitive. You are not mixing because you want to; you are mixing because not mixing would reveal information that the opponent could use to improve their own play.

The practical technique for finding mixed Nash:

1. Guess a support for each player.
2. Write indifference equations: each pure in the support yields the same expected payoff against the opponent's mix.
3. Solve the linear system for the mixing probabilities.
4. Check that the resulting probabilities are in $[0, 1]$ and sum to 1.
5. Check that out-of-support pure strategies do not exceed the equilibrium payoff.
6. If any check fails, try a different support.

This loop is the mixed-equilibrium solver for small games by hand. For large games it is infeasible (exponential in number of supports), and algorithms like Lemke–Howson or CFR are used instead.

Structural Role

Mixed Nash generalizes pure Nash and is preserved under existence theorems:

- **2.2 ⇒ 2.3.** Pure is a special case of mixed.
- **2.3 ⇒ 2.4.** Nash existence applies to the mixed-strategy simplex.
- **2.3 ⇒ 2.5.** Mixed Nash implies rationalizability (support of any equilibrium mix survives iterated best response).
- **2.3 ⇒ 2.6.** Mixed Nash is a special case of correlated equilibrium (with the correlating device being trivial).
- **2.3 ⇒ 9.1.** Fictitious play converges to mixed Nash in zero-sum and some potential games.
- **2.3 ⇒ 9.3.** CFR converges to mixed Nash of the extensive-form game.

Mixed Nash is also the refinement-resistant solution concept: many equilibrium refinements in Module 3 and 4 preserve mixed equilibria while eliminating some pure equilibria that rely on incredible threats.

Mathematical Development

Support enumeration

For a 2×2 bimatrix game (A, B):

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Assume a completely mixed equilibrium $(p, 1-p)$ for Row, $(q, 1-q)$ for Column.

Row's indifference (makes Column willing to mix): Column's payoff from L is $b_{11}p + b_{21}(1-p)$; from R is $b_{12}p + b_{22}(1-p)$. Setting equal:

$$p = \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}}.$$

Column's indifference (makes Row willing to mix):

$$q = \frac{a_{22} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}}.$$

These formulas hold whenever the denominators are non-zero and the resulting p, q lie in $(0, 1)$.

Support guessing for larger games

For $m \times n$ games, one enumerates possible supports. The total number of supports is $(2^m - 1)(2^n - 1)$, exponential in m, n. Most supports do not yield equilibria. Heuristics like Lemke–Howson pivoting systematically explore the support graph.

Oddness theorem

Theorem (Wilson, 1971). Almost every finite game has an odd number of Nash equilibria. "Almost every" means that the set of games with even or infinite equilibria has measure zero in the space of payoff matrices.

This is a topological counting result; it says that Nash equilibria come in odd numbers generically because they are intersections of best-response curves with generic transversality.

Continuity of equilibrium correspondence

The Nash equilibrium correspondence $G \mapsto \{\sigma^* : \sigma^* \text{ is Nash of } G\}$ is upper-hemicontinuous in G on the space of bimatrices: small changes in payoffs produce small changes in the equilibrium set. This is the basis of the robustness and purification arguments of Harsanyi.

Computing mixed Nash for specific game classes

- **Zero-sum games.** LP; polynomial time.
- **Potential games.** Maximize the potential; polynomial in the number of profiles.
- **Two-player general-sum.** Lemke–Howson; exponential worst case, polynomial in practice.
- **n-player general-sum.** PPAD-complete; no known polynomial algorithm.

Worked Example

Battle of the Sexes (mixed)

	O	F
O	$(2, 1)$	$(0, 0)$
F	$(0, 0)$	$(1, 2)$

Pure Nash: (O, O) and (F, F).

Mixed Nash: Guess support $= \{O, F\}$ for both. Row plays $(p, 1-p)$, Column plays $(q, 1-q)$.

Column's indifference (Row's mix makes Column's O vs F equal): $1 \cdot p + 0 \cdot (1-p) = 0 \cdot p + 2 \cdot (1-p) \Rightarrow p = 2(1-p) \Rightarrow p = 2/3$.

Row's indifference: $2q + 0 = 0 + 1(1-q) \Rightarrow q = 1/3$.

Equilibrium: Row plays $(2/3, 1/3)$, Column plays $(1/3, 2/3)$.

Equilibrium payoff. Row: $2 \cdot 1/3 + 0 \cdot 2/3 = 2/3$. Column: $1 \cdot 1/3 + 0 = 1/3\ldots$ wait, let me redo. Row's expected payoff: $pq \cdot 2 + p(1-q) \cdot 0 + (1-p)q \cdot 0 + (1-p)(1-q) \cdot 1 = 2 \cdot (2/3)(1/3) + (1/3)(2/3) = 4/9 + 2/9 = 6/9 = 2/3$. Column's: $1 \cdot (2/3)(1/3) + 2 \cdot (1/3)(2/3) = 2/9 + 4/9 = 6/9 = 2/3$. Both get $2/3$, same payoff, much worse than either pure equilibrium. Mixed Nash is Pareto-dominated.

A 3-strategy game

	L	C	R
T	$(0,0)$	$(2,3)$	$(3,2)$
M	$(3,2)$	$(0,0)$	$(2,3)$
B	$(2,3)$	$(3,2)$	$(0,0)$

This is a 3-strategy "cyclic" game. No pure Nash (check best responses cycle). Completely mixed symmetric equilibrium: guess all three actions in support.

Row's indifference: $U_R(T,\sigma_C) = U_R(M,\sigma_C) = U_R(B,\sigma_C)$. Let $\sigma_C = (q_L, q_C, q_R)$. Row T: $0 \cdot q_L + 2q_C + 3q_R$. Row M: $3q_L + 0 + 2q_R$. Row B: $2q_L + 3q_C + 0$. Setting equal with $q_L + q_C + q_R = 1$ gives a linear system. By symmetry, $q_L = q_C = q_R = 1/3$, and each row's expected payoff is $(0+2+3)/3 = 5/3$ for T, which should match $(3+0+2)/3 = 5/3$ for M, and $(2+3+0)/3 = 5/3$ for B. All equal. Confirmed.

Symmetric completely mixed equilibrium: $\sigma^* = ((1/3,1/3,1/3),(1/3,1/3,1/3))$, each with expected payoff $5/3$.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Mixed equilibrium σ^*	Frequency-based GTO solutions — "bet 62%, check 38%" on every mixed information set
Support of σ_i^*	The set of actions the solver ever takes at a given hand in a given spot
Indifference condition	All actions in a mix must have the same EV against opponent's equilibrium response
Solving for mixed probabilities	Exactly what a solver does iteratively: update frequencies to enforce opponent indifference
Out-of-support pure strategies	Actions the solver never recommends — dominated or dominated-in-context
Guessing supports	What a human analyst does when trying to "reconstruct" a solver's output by hand

Concrete situation

River decision, single-raised heads-up pot. Board: $A\spadesuit K\clubsuit 7\diamondsuit 3\heartsuit 2\clubsuit$. BTN has bet flop and turn, BB has called both. On the river, BTN faces a 2/3-pot bet from BB.

What is the solver doing? With each hand in BTN's range, the solver is computing the call/fold mix that makes BB indifferent between value-betting with their best hands and bluffing with their worst. The indifference condition for BB is:

$$\alpha \cdot (\text{EV of value bet}) + (1-\alpha) \cdot (\text{EV of bluff}) = \text{EV of checking},$$

where α is the fraction of BB's river betting range that is value. The mixed strategy BTN plays with bluff-catchers must make this equation hold; otherwise BB could profitably shift toward more bluffs or more value.

Concrete mix. Suppose BTN has a bluff-catcher like $K\heartsuit T\clubsuit$. The solver outputs: call 65%, fold 35%. These are *not* BTN's preferences — BTN is indifferent between the two (both yield the same EV against

BB's equilibrium betting frequency). The 65/35 is the unique mix at which BB's best response is neither "bet more bluffs" nor "bet fewer bluffs" — it is exactly the frequency that makes BB's own indifference hold.

This is the support characterization in its most operational form: the support is $\{\text{call}, \text{fold}\}$, the indifference condition on *BTN's* side is EV(call) = EV(fold), and the mix is chosen to balance *BB's* incentives.

What this understanding buys you

Mixing is for the opponent, not for you. You mix at a river spot because not mixing would let the opponent shift their own range to exploit you. You do not mix because you are "uncertain" — you mix because the fixed-point condition demands it.

Any deviation from the mix is exploitable — but only if the opponent notices. Against a strong opponent, stick to the mix. Against a weak opponent who will not adjust, deviate toward the exploit. The mixed equilibrium is the unexploitable floor; the best response is the exploit.

When solver output confuses you, run the indifference check yourself. Write down the EV of each action in the mix against the opponent's equilibrium response. They should be equal (up to numerical precision of the solver). If they are, the mix is right. If they are not, the solver has not converged yet.

Cross-Links

- **1.2 Mixed strategies** — the lift that makes mixed equilibria well-defined.
- **1.4 Best response** — mixed best response = support indifference.
- **2.1 Nash equilibrium** — the parent definition.
- **2.2 Pure Nash** — the special case with singleton supports.
- **2.4 Nash existence** — guarantees at least one mixed Nash.
- **2.5 Rationalizability** — broader solution concept.
- **9.1 Fictitious play** — dynamic convergence to mixed Nash.
- **9.3 CFR** — iterative construction of mixed equilibria in extensive form.

Structural Compression

Invariant: In a mixed Nash equilibrium, each player's mixing distribution is supported on pure strategies that are all optimal against the others' equilibrium profile, and these supports satisfy an indifference principle that makes every opponent's equilibrium response a best response in turn.

Mechanism: Linear indifference equations over mixing probabilities, supplemented by inequality constraints verifying out-of-support optimality; existence is guaranteed by convexity of the mixed-strategy simplex and the fixed-point theorem (2.4).

Cross-System Echo: Mixed strategies at equilibrium are the strategic analogue of *stationary distributions* in ergodic Markov chains: the fixed distribution that the system occupies in the long run. The indifference condition is the analogue of detailed balance.

Exercises

1. Compute the mixed Nash of Matching Pennies and verify directly that both players are indifferent at the mix.

2. For the game

	L	R
T	$(3,2)$	$(0,0)$
B	$(0,0)$	$(2,3)$

,

find the pure and mixed equilibria.

3. Apply the indifference principle to compute the mixed equilibrium of a generic 2×2 zero-sum game $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d \neq b + c$. Derive the value of the game.

4. Prove: in any completely mixed Nash equilibrium of a finite game, the number of equality constraints per player equals $|S_i| - 1$, giving a system that may or may not have solutions with non-negative probabilities.

5. State the oddness theorem and show by example that a 2×2 game with infinitely many Nash equilibria is non-generic.

6. Given a bimatrix game with a mixed Nash (p^*, q^*), show that the equilibrium payoff to Row equals $p^{*\top} A q^*$, and that this equals $A_{i,j}$ for any $i \in \mathrm{supp}(p^*)$ and $j \in \mathrm{supp}(q^*)$ when payoff entries coincide.

7. **Argue, structurally,** why the indifference principle — the fact that a mixing player is indifferent between supported pure strategies — is not a contradiction with rationality, even though the player has "no reason" to pick one pure strategy over another. Frame your answer in terms of the self-referential fixed-point nature of Nash equilibrium.

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Concepts: nash-equilibrium, expectation-value, probability-space
Prerequisites: 1.2, 1.4, 2.1, 2.2
Enables: 2.4, 2.5, 9.3

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