Nash's Existence Theorem

Game Theory · Module 2 — Nash Equilibrium

Node 2.4

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Nash Equilibrium **Node:** 2.4

Nash's existence theorem (1950) is the foundational result of the entire field. It says: every finite strategic-form game has at least one Nash equilibrium in mixed strategies. Without this theorem, equilibrium reasoning would be a guess-and-check exercise with no guarantee of finding anything. With it, the solution concept of Nash equilibrium is *universal* — applicable to every game you might write down — and all subsequent developments in the course proceed knowing that the object they are studying exists. The proof is a beautiful application of Kakutani's fixed-point theorem, generalizing the minimax theorem (1.5) from zero-sum to general-sum games and from bilinear to multilinear payoffs.

Formal Statement

Nash's Theorem (1950). Every finite strategic-form game $G = (N, \{S_i\}, \{u_i\})$ has at least one Nash equilibrium in mixed strategies.

More generally, the theorem holds for any game where:

1. The strategy spaces $\Delta(S_i)$ (or, in continuous games, X_i) are non-empty, compact, and convex subsets of Euclidean space.
2. The payoff functions U_i are continuous in the full profile σ and quasi-concave in own strategy σ_i (holding others fixed).

Under these conditions, a Nash equilibrium exists. For finite games, the multilinear extension automatically satisfies continuity and quasi-concavity (linearity is the special case).

Corollary. Every finite two-player zero-sum game has a value (minimax theorem, Node 1.5) and this value is attained by a pair of Nash equilibrium strategies.

Intuition

Nash's theorem is a "fixed point exists" result. Every player has a best-response correspondence BR_i from the opponent's profile to the set of optimal own-strategies. The joint best-response $\mathrm{BR}(\sigma) = \mathrm{BR}_1(\sigma_{-1}) \times \cdots \times \mathrm{BR}_n(\sigma_{-n})$ is a correspondence from the product of simplices to itself. A fixed point $\sigma^* \in \mathrm{BR}(\sigma^*)$ is a Nash equilibrium.

Kakutani's theorem says: a correspondence from a non-empty, compact, convex set to itself, with non-empty, convex, compact values and upper-hemicontinuous graph, has a fixed point. Each of these properties of BR is derivable from the multilinearity and continuity of U_i, so the theorem applies, and a fixed point exists. That fixed point is a Nash equilibrium.

The conceptual work Nash did was *identifying* the correct setup: map best-response to a correspondence on the simplex, verify the Kakutani hypotheses, conclude existence. The technical work is mostly packaged inside Kakutani. The result is the single most-cited theorem in game theory and the reason Nash won the Nobel Prize in Economics in 1994.

Structural Role

Nash existence is the *launching pad* for every subsequent chapter. Without it:

- **2.5 Rationalizability** would have no guaranteed survivors.
- **3.5 Subgame perfection** would be a refinement of a potentially empty set.
- **4.3 Bayesian Nash** would need its own existence theorem.
- **4.4 Perfect Bayesian equilibrium** would not be guaranteed.
- **7.2 ESS** as a Nash refinement would have no base object.
- **9 Learning** algorithms would have no target fixed point.

Every result that characterizes or refines Nash equilibrium assumes existence. Every computational method for "finding" Nash equilibrium proceeds as if such an object exists. Nash's theorem is what makes that assumption legitimate.

The theorem also generalizes in several directions:

- **Glicksberg (1952):** continuous games on compact metric spaces with continuous payoffs.
- **Debreu (1952), Fan (1952):** games with concave payoffs on compact convex sets.
- **Mertens (1986):** universal belief spaces, Bayesian games with incomplete information.

These generalizations preserve the structural template: non-empty compact convex domain + continuous quasi-concave payoffs + fixed-point theorem.

Mathematical Development

Kakutani's fixed-point theorem

Theorem (Kakutani, 1941). Let $X \subseteq \mathbb{R}^n$ be non-empty, compact, and convex. Let $\phi : X \rightrightarrows X$ be a correspondence with: - $\phi(x)$ non-empty, compact, and convex for every $x \in X$. - Graph $\{(x, y) : y \in \phi(x)\}$ closed (equivalently, ϕ is upper-hemicontinuous).

Then ϕ has a fixed point: there exists $x^* \in X$ with $x^* \in \phi(x^*)$.

Kakutani's theorem generalizes Brouwer's fixed-point theorem (for continuous functions) to correspondences. It is itself a corollary of Brouwer via an approximation argument.

Proof of Nash's theorem

Let $X = \prod_i \Delta(S_i)$. Check: - X **is non-empty, compact, convex.** Each simplex $\Delta(S_i)$ is; products preserve these properties. - BR **is non-empty-valued.** U_i is continuous on the compact domain; the Weierstrass theorem gives a maximizer. - BR **is convex-valued.** If $\sigma_i, \sigma_i' \in \mathrm{BR}_i(\sigma_{-i})$, then by multilinearity of U_i, any convex combination is also in $\mathrm{BR}_i(\sigma_{-i})$. - BR **has closed graph.** By the Berge Maximum Theorem (or direct verification), BR_i is upper-hemicontinuous because the objective is continuous in both arguments and the constraint set is constant.

All Kakutani hypotheses hold. Apply Kakutani: there exists $\sigma^* \in X$ with $\sigma^* \in \mathrm{BR}(\sigma^*)$. By definition, σ^* is a Nash equilibrium. ■

Alternate proof via Brouwer

Define a continuous self-map $f : X \to X$ by

$$f_i(\sigma)(s_i) = \frac{\sigma_i(s_i) + \max(0, U_i(s_i, \sigma_{-i}) - U_i(\sigma))}{1 + \sum_{s_i' \in S_i} \max(0, U_i(s_i', \sigma_{-i}) - U_i(\sigma))}.$$

This map "adjusts" each player's strategy toward pure actions whose payoffs exceed the current expected payoff — a "gain-function" increment. f is continuous on the compact convex X and so has a Brouwer fixed point. At a fixed point, the adjustment terms must vanish for every s_i, which forces $U_i(s_i, \sigma_{-i}) \leq U_i(\sigma)$ for all s_i — exactly the Nash condition.

This is Nash's original (1951) proof, simpler than Kakutani but more ad hoc.

Why multilinearity matters

The proof uses multilinearity of U_i in two places: 1. To show BR_i is convex-valued (combinations of best-responses are best-responses). 2. To show U_i is continuous on the simplex (multilinear is automatically continuous).

Without multilinearity, one would need alternative convexity/concavity hypotheses.

Why compactness matters

Compactness gives the Weierstrass theorem (continuous function on compact set attains its max). Without compactness, even single-player decision problems may fail to have optimal strategies. In continuous games on non-compact spaces, one compactifies or truncates before applying a fixed-point theorem.

Worked Example

Verifying all Kakutani hypotheses on Matching Pennies

$$A = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}.$$

Strategy spaces: Row plays $\sigma_1 = (p, 1-p)$, Column plays $\sigma_2 = (q, 1-q)$, $p, q \in [0, 1]$.

- $X = [0, 1]^2$**.** Non-empty, compact, convex. ✓
- $U_1(p, q) = p(2q-1) + (1-p)(1-2q) = (2q-1)(2p-1)$**.** Multilinear in p and q. ✓
- $\mathrm{BR}_1(q)$**:** $\{1\}$ if $q > 1/2$; $[0, 1]$ if $q = 1/2$; $\{0\}$ if $q < 1/2$. Non-empty, compact, convex. ✓
- **Graph of** BR_1 **is closed:** yes (the jump at $q = 1/2$ is filled in by $[0, 1]$). ✓

By Kakutani, a fixed point exists. Indeed it does: $(1/2, 1/2)$, which is a fixed point because at $q = 1/2$, $\mathrm{BR}_1 = [0, 1]$ contains $1/2$, and symmetrically. Nash equilibrium confirmed.

A three-player game

Three players each choose $\{A, B\}$. Payoff equals $+1$ if all three agree, -1 otherwise. This is a coordination game with two pure Nash equilibria (AAA and BBB) and one symmetric mixed Nash at $(1/2, 1/2, 1/2)$ — three equilibria total. Kakutani guarantees at least one; we have found three. The oddness theorem (2.3) suggests generically there are an odd number.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Kakutani fixed point	GTO equilibrium of poker — guaranteed to exist in every finite abstraction
Compactness of $\Delta(S_i)$	Finite abstraction of a poker information set — a simplex over a bounded number of actions
Convexity of BR_i	At an indifference point, any mix over tied actions is a best response
Upper hemicontinuity	Small changes in opponent frequencies produce small changes in best responses
Nash's theorem's role	Justifies the existence of "the GTO solution" for any finite poker abstraction before we try to compute it

Concrete situation

Why Cepheus could claim to "solve" HU limit hold'em. Before the 2015 CFR run, it was mathematically certain that a Nash equilibrium of HULHE exists, because HULHE is a finite game (though very large: ~10^14 information sets) and Nash's theorem applies. The existence was not in doubt; the question was computing it.

CFR is an algorithm that converges to an ϵ-Nash equilibrium as the number of iterations grows. Cepheus ran for several months on a cluster and converged to an ϵ so small (one-millibet per game) that the resulting strategy is "essentially unbeatable" in the statistical sense: the entire game value achievable by any opponent is bounded above by ϵ.

Without Nash's theorem, the claim "there is a GTO strategy" would have been conjectural. With Nash's theorem, it was a *theorem* — Cepheus was computing an object whose existence was established 65 years earlier.

What this understanding buys you

GTO exists. Before any computation, the theorem says so. This is not a trivial point. Early poker AI research briefly entertained the notion that the game might be too large to have a "solution" in any meaningful sense. Nash's theorem rules this out: the space of mixed strategies on any finite abstraction is a product of simplices, compact and convex, and the best-response correspondence on it has a fixed point. The fixed point is the GTO strategy.

Finite abstraction matters. Nash's theorem applies to finite games. When a solver abstracts the infinite HUNL tree (e.g., by bucketing bet sizes), the abstraction is finite, and Nash's theorem guarantees a fixed point of the abstracted game. The abstraction error — how much the abstracted Nash differs from the true HUNL Nash — is a separate (and hard) problem, but existence of the abstracted solution is not in doubt.

Continuous-action poker variants. Some toy analyses (e.g., AKQ games, Kuhn poker) use continuous bet sizes. Nash existence extends to such games via Glicksberg's theorem, provided payoffs are continuous and strategy spaces are compact. The theoretical guarantee carries over.

Cross-Links

- **1.2 Mixed strategies** — the simplex domain.
- **1.4 Best-response correspondences** — the operator.
- **1.5 Zero-sum minimax** — the two-player zero-sum case with a simpler LP proof.

- **2.1 Nash equilibrium** — the object the theorem guarantees.
- **2.3 Mixed strategy Nash** — the form the equilibrium takes.
- **Glicksberg's theorem** — continuous-action extension.
- **Real analysis — compactness, continuity, Brouwer/Kakutani** — the mathematical backbone.
- **Systems thinking — fixed points** — the unifying concept.

Structural Compression

Invariant: Every finite game has at least one Nash equilibrium in mixed strategies, guaranteed by the existence of a fixed point of the joint best-response correspondence on the compact convex mixed-strategy product space.

Mechanism: Kakutani's fixed-point theorem applied to a correspondence with the required regularity properties (non-empty, convex, compact values; closed graph), derived from continuity and multilinearity of expected payoffs.

Cross-System Echo: Fixed-point theorems are the universal technique for proving existence of equilibria across mathematical sciences: Brouwer for maps, Kakutani for correspondences, Banach for contractions, Lefschetz for topological equilibria, Tarski for order-preserving maps on lattices. Nash existence is the strategic instance of a pattern that pervades mathematics.

Exercises

1. Verify the four Kakutani hypotheses on BR directly for the Prisoner's Dilemma (mixed-strategy version).
2. Prove that the best-response correspondence in a finite game is upper-hemicontinuous. (Use the Berge Maximum Theorem.)
3. Show that multilinearity of U_i in σ_i implies convexity of $\mathrm{BR}_i(\sigma_{-i})$.
4. Sketch the construction of Nash's original Brouwer-based proof, filling in the definition of the map f and verifying it is continuous.
5. Give a counterexample to Nash existence in an infinite game with non-compact strategy spaces. (E.g., players choose a real number; the payoff depends on the difference.)
6. Show that Glicksberg's theorem extends Nash existence to continuous games on compact metric spaces with continuous payoffs. What replaces multilinearity?
7. **Argue, structurally,** why Nash's theorem is a consequence of topology rather than algebra — that is, why the hypotheses are topological (compactness, continuity) rather than arithmetic (specific functional forms). What would be the analog of Nash's theorem for infinite-player games or games with non-metrizable strategy spaces?

Game Theory · Module 2 — Nash Equilibrium · Node 2.4
Concepts: nash-equilibrium, fixed-point, convexity
Prerequisites: 1.2, 1.4, 2.1, 2.3
Enables: 2.5, 2.6, 3.5, 4.3, 7.2

hbar.university — own.your.cognition