Correlated Equilibrium

Game Theory · Module 2 — Nash Equilibrium

Node 2.6

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Nash Equilibrium **Node:** 2.6

Correlated equilibrium (Aumann, 1974) is a generalization of Nash equilibrium that allows players to condition their actions on a shared public signal. It is a strictly broader concept than Nash: every Nash equilibrium is a correlated equilibrium (take the signal to be degenerate), but there are correlated equilibria that are not the mixture of any Nash. It often supports *better* outcomes than any Nash — higher social welfare, more cooperation — and is the solution concept that naturally arises in no-regret learning (Hart–Mas-Colell), making it the central target of learning dynamics (Node 9.2). It is also computationally tractable: correlated equilibria can be found in polynomial time via linear programming, unlike Nash equilibria.

Formal Definition

Let $G = (N, \{S_i\}, \{u_i\})$ be a finite game.

A **correlated equilibrium** is a probability distribution $\mu \in \Delta(S)$ over strategy profiles such that for every player i, every $s_i \in S_i$ with $\mu(s_i) > 0$, and every alternative action $s_i' \in S_i$:

$$\sum_{s_{-i} \in S_{-i}} \mu(s_{-i} \mid s_i)\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \mu(s_{-i} \mid s_i)\, u_i(s_i', s_{-i}).$$

Interpretation. A trusted external mediator draws a profile $s \sim \mu$ and tells each player i only their own component s_i. Given this "recommendation," player i updates their belief about the others' actions to $\mu(\cdot \mid s_i)$, and the condition says that following the recommendation s_i is a best response under this updated belief.

Equivalent obedience formulation: for every s_i and every function $f : S_i \to S_i$,

$$\sum_s \mu(s) u_i(s) \geq \sum_s \mu(s) u_i(f(s_i), s_{-i}).$$

That is, no deviation rule f is profitable — players have no incentive to deviate from any recommended action.

Every Nash equilibrium is a correlated equilibrium. Take $\mu = \sigma_1^* \otimes \cdots \otimes \sigma_n^*$ (product distribution).

Intuition

Imagine a trusted mediator draws a profile from a distribution μ and whispers to each player what they should play. You only hear your own recommendation. Given what you heard, you form a conditional belief about what the others were told, and ask: is my recommendation a best response under this belief?

If the answer is yes for every player and every possible recommendation they could receive, the distribution μ is a correlated equilibrium. No player benefits from disobeying their recommendation.

The correlation is the key: the mediator's draws can be *dependent* across players. A standard traffic light is a correlated equilibrium in a "chicken" game: the mediator (the light) tells one driver to go and the other to stop, with perfect negative correlation. Obeying the light is a best response given what you see. This cannot be achieved by a product distribution (= Nash in mixed strategies), because independent mixing would sometimes tell both drivers to go.

Correlated equilibrium expands the solution space by introducing a **correlation device**. It can strictly improve social welfare over the best Nash equilibrium. And it is achievable in practice by any mechanism that can broadcast a public signal (a traffic light, a lottery, a shared randomness source).

Structural Role

Correlated equilibrium is the natural solution concept for settings with shared randomness:

- **2.1 Nash ⇒ 2.6 Correlated equilibrium.** Every Nash is correlated.
- **2.6 ⇒ 9.2 No-regret learning.** Time-averaged play under no-regret algorithms converges to the set of correlated equilibria, not necessarily to Nash.
- **2.6 ⇒ Hart–Mas-Colell theorem.** Individual no-internal-regret dynamics converge to correlated equilibrium.
- **Computational tractability.** Finding a correlated equilibrium is a polynomial-time LP, whereas Nash is PPAD-complete. This makes correlated equilibrium the "computer-feasible" solution concept.
- **Welfare.** Correlated equilibria can strictly Pareto-dominate all Nash equilibria in a game (example: Chicken).

In mechanism design (Module 6), correlated equilibrium arises when the designer provides a correlation device (a lottery or a shared random message). In repeated games (Module 5), the correlation device emerges endogenously from past history.

Mathematical Development

LP formulation

For a finite game with n players and strategy profiles in S, a correlated equilibrium is a distribution $\mu \in \Delta(S)$ satisfying, for every player i and every pair (s_i, s_i') with $s_i \neq s_i'$:

$$\sum_{s_{-i}} \mu(s_i, s_{-i}) \left[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\right] \geq 0.$$

This is a linear constraint in μ. Together with $\mu \geq 0$ and $\sum_s \mu(s) = 1$, it defines a polytope. Correlated equilibria are exactly the points in this polytope. Finding one is a feasibility LP — solvable in polynomial time.

Chicken example

	D	C
D	$(0,0)$	$(4,1)$
C	$(1,4)$	$(3,3)$

Nash equilibria: (D, C) and (C, D) pure, plus a mixed one with payoff 8/3 each.

Correlated equilibrium via public signal. Mediator draws one of three profiles uniformly: $(D, C), (C, D), (C, C)$, each with probability 1/3. Tell each player only their own action.

Check player 1's obedience. If told "D," player 1 knows the profile is (D, C) (the only one with D for player 1 — wait, we should check). Actually the profiles are $(D, C), (C, D), (C, C)$. Player 1's action is D only in (D, C). If player 1 is told D, they infer the profile is (D, C), and their expected payoff from obeying is 4 (vs deviation to C giving 3). Obey.

If player 1 is told C, they infer the profile is either (C, D) or (C, C) with equal probability (each conditional probability 1/2). Expected payoff from obeying: $0.5 \cdot 1 + 0.5 \cdot 3 = 2$. From deviating to D: $0.5 \cdot 0 + 0.5 \cdot 4 = 2$. Tie — obeying is weakly optimal. ✓ (In a strict sense, need $\geq$, which holds.)

Player 2's obedience: symmetric. ✓

Welfare comparison. Expected payoff: player 1 gets $1/3(4+1+3) = 8/3$, player 2 gets $1/3(1+4+3) = 8/3$, same as mixed Nash. Player 1+2 sum: $16/3 \approx 5.33$.

Better correlated equilibrium. Try $(D, C), (C, D), (C, C), (C, C)$ with probabilities $1/4, 1/4, 1/2$ — check. Expected payoff: player 1 gets $0.25 \cdot 4 + 0.25 \cdot 1 + 0.5 \cdot 3 = 2.75$. Better than $8/3 \approx 2.67$. Verify obedience: conditioning on D, inference is (D, C), expected payoff 4 vs deviation 3 — obey. Conditioning on C, inference is (C, D) with probability 1/3, (C, C) with probability 2/3. Expected payoff from obeying: $1/3 \cdot 1 + 2/3 \cdot 3 = 7/3$. From deviating to D: $1/3 \cdot 0 + 2/3 \cdot 4 = 8/3$. $8/3 > 7/3$ — NOT obedient. This distribution is not a correlated equilibrium.

Correlated equilibrium has structural constraints; not every seemingly good distribution works.

Hart–Mas-Colell theorem

Theorem (Hart–Mas-Colell, 2000). If each player in a repeated game uses a regret-matching algorithm, then the empirical distribution of play converges almost surely to the set of correlated equilibria.

Regret-matching is a simple adaptive rule: the probability of playing action s_i tomorrow is proportional to the total regret (if positive) of not having played s_i in the past. No knowledge of the game or opponents' payoffs is required, only own-payoff feedback.

Polynomial-time computation

Correlated equilibrium is an LP; Nash is PPAD-complete. The gap between these two computational classes is one of the major structural differences between the two solution concepts. In large games, "find a correlated equilibrium" is feasible; "find a Nash equilibrium" may not be.

Worked Example

Coordination game with a traffic signal

Two drivers meet at an intersection. Actions: Go, Stop.

	Go	Stop
Go	$(-10, -10)$	$(1, 0)$
Stop	$(0, 1)$	$(-1, -1)$

Two pure Nash equilibria: $(\text{Go}, \text{Stop}), (\text{Stop}, \text{Go})$. Mixed Nash: symmetric, with probability of Go $= 1/10$ (indifference calculation).

Correlated equilibrium via traffic light. Mediator tosses a fair coin: heads $\to$ (Go, Stop), tails $\to$ (Stop, Go). Each player is told only their own recommendation. Player 1's expected payoff from obeying

"Go" is 1 vs deviation to Stop giving -1 — obey. From obeying "Stop": 0 vs deviation to Go giving -10 — obey. Similarly for player 2. Correlated equilibrium verified.

Expected payoff: $1/2 \cdot 1 + 1/2 \cdot 0 = 1/2$ each. Vastly better than the mixed Nash (which has frequent $(\mathrm{Go}, \mathrm{Go})$ crashes) and equal to the average of the two pure Nash.

This is precisely what traffic lights do. The light is the mediator; the "red means stop, green means go" convention is the obedience rule; the independence of the draw from driver identity makes the mechanism fair.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Correlation device	Rare in heads-up cash poker; more relevant in staking networks and multi-way tournaments
Correlated equilibrium	Outcome distribution where obeying the mediator is a best response
Correlation via community cards	Not a correlation device in the technical sense — the cards are public information
Poker bot collusion	A genuine correlation device: two bots sharing information is a correlated mechanism, and the resulting play may be a correlated equilibrium of the multi-way game
No-regret $\to$ correlated equilibrium	CFR and related algorithms converge to correlated equilibria in multi-player games, not to Nash

Concrete situation

Pluribus and 6-max. When Pluribus (Facebook/CMU 2019) solved 6-max no-limit hold'em, it did not, strictly speaking, compute a Nash equilibrium. For $n \geq 3$, Nash equilibrium in extensive-form games is both hard to define cleanly (multiple equilibria with welfare tradeoffs) and computationally PPAD-complete. What Pluribus computed was a strategy that performs well in a no-regret sense — and the Hart–Mas-Colell theorem tells us such strategies converge to *correlated* equilibria of the game, not Nash.

In practice, Pluribus's play was close enough to "nice" correlated equilibria that it dominated top human professionals in play — but there is no theorem saying it plays Nash in the 6-max game. There is a theorem saying it plays a correlated equilibrium asymptotically (modulo subtleties of the abstraction and online blueprint).

This is not a defect; it is the natural solution concept for multi-player games with limited computational resources. Correlated equilibria have higher welfare potential and are computationally easier. For multi-way poker, correlated equilibrium is the right object.

What this understanding buys you

For 2-player poker (HU), GTO = Nash = correlated equilibrium. The simplex of correlated equilibria in a zero-sum two-player game coincides with the Nash set (all live at the minimax value). You do not need correlated equilibrium as a distinct concept for HU.

For 3+ player poker, GTO as Nash becomes ambiguous. There are multiple Nash equilibria in 6-max, with different welfare properties, and the strategies your opponents actually play may not coincide with any specific Nash. Correlated equilibrium is broader and captures more of what "good play" actually looks like in practice. The equilibria Pluribus converges to are correlated, not Nash.

Collusion in poker is formal collusion on a correlation device. Two colluders sharing information via a side channel are implementing a correlation device; the resulting play is a correlated equilibrium of the game *with the side-channel mediator*. This is why it is illegal: correlated equilibrium of the mediated game gives strictly higher welfare to the colluders than Nash of the unmediated game.

Cross-Links

- **2.1 Nash equilibrium** — correlated generalizes Nash.
- **2.3 Mixed Nash** — Nash is the product-distribution special case.
- **2.5 Rationalizability** — correlated CE is incomparable to rationalizability.
- **5 Repeated games** — history serves as an endogenous correlation device.
- **6 Mechanism design** — external correlation devices as mechanisms.
- **9.2 No-regret learning** — dynamics converging to correlated equilibrium.
- **Linear programming** — the algorithmic backbone.

Structural Compression

Invariant: A correlated equilibrium is a joint distribution over profiles such that, conditional on one's own recommendation, obeying the mediator is a best response. The set of correlated equilibria is convex, contains all Nash equilibria, and can strictly improve welfare.

Mechanism: Conditional-expectation best-response constraints, encoded as linear inequalities on the distribution $\mu \in \Delta(S)$; feasibility is polynomial-time.

Cross-System Echo: Correlated equilibrium captures the structural role of *signaling* and *institutional coordination* across systems: traffic lights, auctions with reserve mechanisms, coordinating protocols in distributed systems, shared-randomness communication channels. Whenever a system introduces a common signal, it lifts the equilibrium concept from Nash to correlated.

Exercises

1. For the Chicken game, find a correlated equilibrium with strictly higher expected welfare than the mixed Nash.
2. Write out the full LP for computing correlated equilibrium of a 3×3 coordination game.
3. Prove that every Nash equilibrium is a correlated equilibrium (product-distribution case).
4. Show that the set of correlated equilibria is convex in $\Delta(S)$.
5. Construct a 2×2 game in which the set of correlated equilibrium payoffs strictly contains the set of Nash equilibrium payoffs.
6. State the Hart–Mas-Colell theorem and explain why regret-matching is only guaranteed to converge to correlated, not Nash, equilibria.
7. **Argue, structurally,** why the computational gap between Nash (PPAD) and correlated (LP) suggests that correlated equilibrium is a "more natural" target for learning algorithms in large games. What does this imply about the practical meaning of "solving" a game?

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Concepts: nash-equilibrium, conditional-probability, incentive-compatibility

Prerequisites: 2.1, 2.3
Enables: 4.3, 9.2

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