Information Sets and Imperfect Information

Game Theory · Module 3 — Extensive Form Games

Node 3.2

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Extensive Form Games **Node:** 3.2

An information set is a partition block of decision nodes that the acting player cannot tell apart. It is the formal device that encodes "what the player knows when they move." Without information sets, extensive-form games would be restricted to perfect information — no hidden cards, no simultaneous moves, no private types — and the whole of poker, sealed-bid auctions, and signaling would be out of reach. This lesson is where the game-theoretic treatment of *uncertainty about the state of the game* enters, and it is the structural prerequisite for strategies (3.3), beliefs (4.x), and sequential equilibrium (4.6).

Formal Definition

Let $\Gamma = (N, T, P, A, u)$ be an extensive-form game. For each player i, let $T_i = \{t \in T \setminus Z : P(t) = i\}$ be the set of decision nodes owned by i.

An **information partition** for player i is a partition $\mathcal{I}_i$ of T_i into **information sets** such that:

1. **Same action set within a block.** For every $I \in \mathcal{I}_i$ and every $t, t' \in I$, $A(t) = A(t')$. (The acting player has the same set of available actions at every node in I; otherwise they could tell them apart from the choices they face.)
2. **No node precedes another in the same block.** No node $t \in I$ is an ancestor of any other $t' \in I$. (Otherwise the acting player would know the game had already visited t, and could distinguish.)

An information set with $|I| = 1$ is a **singleton** — the acting player knows exactly where they are. A game with only singleton information sets for every player has **perfect information**. An information set with $|I| > 1$ represents genuine uncertainty: the acting player knows the game is "somewhere in I" but not where.

The game is said to have **perfect recall** if each player, at any information set, remembers everything they knew and did at every previous own-information-set along the path. Formally: for every player i and every $I \in \mathcal{I}_i$, all nodes in I share the same sequence of i's past information sets and chosen actions.

Intuition

An information set is a "fog" over a group of nodes. When you reach the fog, you know you're somewhere in it, but you can't tell where. Since you have the same actions available at each node in the fog, you must choose one action for the whole information set — your action cannot depend on which specific node you're at, because you don't know.

Two canonical ways information sets arise:

Hidden type. A chance node deals you a private card. Subsequent decision nodes that differ only in which card you hold are in the same information set for the opponent (who can't see your card). For you, they are in different information sets (you can see your card). Poker is the quintessential example.

Simultaneous moves. Two players move at "the same time" — that is, neither observes the other's move before making their own. This is modeled as a sequential game where the second mover's decision nodes are all in a single information set (they can't tell which action the first mover took). Rock–Paper–Scissors becomes an extensive-form game: P1 picks, then P2 picks, but P2 has a 3-node information set that they can't distinguish.

Perfect recall is a technical but essential property: it says you don't forget your own past moves or what you knew at them. Kuhn's theorem (1.6) requires perfect recall. Most natural games have it; some abstractions (e.g., poker information-set bucketing) can introduce imperfect recall, which breaks the equivalence of behavioral and mixed strategies and the nice properties of CFR.

Structural Role

Information sets are the bridge between perfect-information games and the full generality of strategic interaction:

- **3.1 ⇒ 3.2.** Information sets refine the tree structure.
- **3.2 ⇒ 3.3.** Strategies are defined on information sets, not nodes. A pure strategy is a function $s_i : \mathcal{I}_i \to A$.
- **3.2 ⇒ 3.5.** Subgame perfection requires that "subgames" respect information sets — a subgame can only be rooted at a singleton information set.
- **3.2 ⇒ 4.1.** Bayesian games introduce type-dependent information sets via an initial chance move.
- **3.2 ⇒ 4.4.** Perfect Bayesian equilibrium adds beliefs — distributions over nodes within an information set — to handle play at non-singleton sets.
- **3.2 ⇒ 9.3.** CFR computes regret at each information set, not at each node.

Information sets are also the reason poker is a game rather than a decision problem: the opponent's hole cards are hidden, creating non-singleton information sets that force strategic reasoning about distributions over the unseen state.

Mathematical Development

Counting information sets

For a player whose decision nodes lie at d depths in the tree, and who can distinguish k_j information states at depth j, the number of information sets is $\sum_j k_j$. Strategies are functions from information sets to actions, so the strategy-space size is $\prod_{I \in \mathcal{I}_i} |A(I)|$, potentially much smaller than the raw node count.

Perfect recall condition

Formally: player i has perfect recall iff for every information set $I \in \mathcal{I}_i$ and every pair $t, t' \in I$, the sequences of i's own information sets and chosen actions along the path from the root to t and to t' are identical.

Perfect recall guarantees Kuhn's theorem: behavioral and mixed strategies yield the same outcome distributions. Without perfect recall, the two can differ: a mixed strategy can correlate actions at "the same" information set visited at different points, which a behavioral strategy cannot.

Simultaneous moves via information sets

To model a two-player simultaneous game in extensive form: P1 moves first (root decision node), then P2 moves. All of P2's decision nodes (one per P1 action) are placed in a single information set — P2 cannot distinguish them, which is what "simultaneity" means. P1's strategy is a single action; P2's strategy is also a single action (because P2 has only one information set). The extensive form reproduces the simultaneous game exactly.

Signaling games

In a signaling game, nature draws a type for the Sender, who then sends a message. The Receiver observes the message but not the type. This creates an information partition for the Receiver: messages that could have come from multiple types put the Receiver's decision nodes into the same information set. Node 4.5 develops signaling games in detail.

Example: simple poker

Two players each ante 1. Nature deals each player a card from $\{H, L\}$ uniformly. P1 sees only their own card, then decides Bet or Fold. If P1 Bets, P2 sees P1's action but not P1's card, then decides Call or Fold. If P2 Calls, cards are revealed and the higher card wins.

- P1's decision nodes: 2 nodes (one per P1's card). Each is its own information set (P1 knows their own card).
- P2's decision nodes: after each combination of (P1's card, P2's card, P1's action), but P2 doesn't see P1's card. P2 has an information set per (P2's card, P1's action = Bet) combination — 2 information sets total (one per P2's card, only reached after Bet).
- Total information sets: 2 for P1, 2 for P2.

Strategy space: P1's pure strategies are $\{B, F\}^2$ — 4 plans. P2's pure strategies are $\{C, F\}^2$ — 4 plans. Total profile space: $4 \times 4 = 16$.

Worked Example

The AKQ game

A classic toy poker game. Two players ante 1 each. Deck has 3 cards $\{A, K, Q\}$; each player gets one. P1 acts first: Bet 1 or Check. If P1 Bets, P2 may Call or Fold; cards revealed, higher card wins. If P1 Checks, cards revealed immediately at showdown.

Information structure. - P1's cards: $\{A, K, Q\}$, each with conditional prob 1/3, and given P1's card, P2's card is uniform on the remaining 2. - P1's information sets: 3 (one per P1's card). - P2's information sets: 3 (one per P2's card), but only reached after a P1 Bet.

Strategies. - P1 pure strategies: $2^3 = 8$ (action at each of 3 P1 information sets). - P2 pure strategies: $2^3 = 8$ (action at each of 3 P2 information sets, all conditional on P1 having bet).

Solution sketch. Equilibrium analysis on this game: P1 bets A always, mixes on Q (as a bluff), checks K; P2 calls with A always, mixes on K, folds Q. The mixing frequencies satisfy the indifference conditions. This is the smallest example where polarized value betting + bluff catching + bluffing are all present, and is the standard "first poker game" in game-theory courses.

Information-set constraint: same actions

Consider a tree where at node t player 1 has actions $\{A, B, C\}$ and at node t' player 1 has actions $\{A, B\}$. These two nodes **cannot** be placed in the same information set for player 1, because the action sets differ. The constraint $A(t) = A(t')$ within an information set is the way the formalism encodes that the player cannot distinguish by "what choices are presented."

In an information set, you know it is your turn and you know your action menu. What you don't know is the full history.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Player's information set	"What you know right now": your hole cards + public board + betting history
Hidden hole cards	Opponent's 2 private cards are unknown to you → opponent's decision points with different hole cards are in the *same* information set from your perspective, different ones from theirs
Singleton information set	Your own decision points — you know your own cards
Non-singleton information set	Opponent's decision points (from your viewpoint)
Perfect recall	"I remember my own cards and past actions" — true in standard poker
Imperfect recall (pathological)	Only arises in abstractions that bucket "similar hands" together, collapsing some distinctions
Information set count (HUNL)	~10^{161} after a reasonable abstraction — the number a solver has to store regret values for

Concrete situation

Flop information set in HUNL. BTN raises preflop, BB calls. Flop comes 9♠7♡2♣. BB checks, BTN must decide a c-bet action.

What is BTN's information set here? BTN knows: (a) their own hole cards, (b) the three flop cards, (c) the full preflop betting history (BTN raise, BB call), (d) BB's check action on the flop. BTN does **not** know BB's hole cards.

BTN's information set is the set of all game histories consistent with this observable information — it contains one node per possible BB hole-card combination, since BTN cannot distinguish between them. There are $\binom{47}{2}$ − card conflicts ≈ 1000+ such nodes, all in the same information set from BTN's perspective.

A pure strategy for BTN at this information set must specify a single action — BTN cannot say "c-bet if BB has AK, check if BB has 72." BTN does not know BB's hand. The strategy is a single action (or a mixed distribution over actions) that applies to all of BB's possible holdings.

But BTN's own range matters: BTN knows their own hole cards, so each of BTN's possible hands has its own separate information set. If BTN holds AA, that is information set 1; if $72o$, that is information set 2. The solver outputs a different mixed strategy at each.

What this understanding buys you

You strategize over information sets, not nodes. When you think "what should I do with AQ on this flop," you are thinking about your strategy at a specific information set. You are not thinking about the specific opponent holding; that is unknown to you. The fog in the tree is exactly the content of strategic uncertainty.

Range vs hand. The distinction between "my range" and "my hand" maps directly to: my range is the distribution over my possible hole cards (the information partition opponents face), and my hand is the specific realization (which selects my own information set). Good players think about their range at every decision because that is what the opponent sees; they take an action from their specific hand's information set because that is what they personally face.

Abstraction and imperfect recall. When solvers abstract information sets (e.g., merging similar board textures to save compute), they risk introducing imperfect recall, which can subtly break optimality. This is one of the deep technical issues in modern poker solving, and it is solvable only via careful subgame resolving (Libratus, DeepStack).

Cross-Links

- **3.1 Game trees** — the structural carrier of information sets.
- **3.3 Strategies in extensive form** — functions on information sets.
- **3.4 Backward induction** — works cleanly only on singleton information sets.
- **3.5 Subgame perfection** — subgames must respect information sets.
- **4.1 Bayesian games** — type chance node generates non-singleton information sets.
- **4.4 Perfect Bayesian equilibrium** — adds beliefs at non-singleton sets.
- **4.6 Sequential equilibrium** — makes beliefs fully consistent.
- **9.3 CFR** — iterates regret at each information set.

Structural Compression

Invariant: An information set is a maximal set of decision nodes indistinguishable to the acting player — containing nodes with the same action menu, none of which is an ancestor of another — and pure strategies assign one action to each information set.

Mechanism: Partition structure on the set of decision nodes owned by each player; the perfect-recall condition ties the partition to the history of own-information and own-actions along each path.

Cross-System Echo: Information sets are the game-theoretic version of "observation histories" in partially observable Markov decision processes (POMDPs). In both settings, the decision-maker's effective state is not the world state but the equivalence class of world states consistent with their observations. Belief states, sufficient statistics, and filtration structures all generalize the information-set construction.

Exercises

1. Draw the extensive-form representation of Matching Pennies using information sets to model simultaneity.
2. Draw the extensive-form representation of the AKQ poker game, labeling each player's information sets.
3. Prove that the "same action set within an information set" axiom is necessary: give a game where violating it produces an ill-defined strategy.
4. Construct a game with perfect recall by giving a path and verifying the condition at each information set along it.
5. Construct a minimal imperfect-recall game and give a mixed strategy whose outcome distribution is not achievable by any behavioral strategy.
6. In a two-stage game where player 1 moves first but player 2 cannot see player 1's move, how many information sets does player 2 have? What is player 2's strategy space?
7. **Argue, structurally,** why information sets are the unique mechanism by which extensive-form games capture both hidden information (private types, hole cards) and simultaneous choice. What feature of tree structure makes this unification possible?

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Concepts: incomplete-information

Prerequisites: 3.1
Enables: 3.3, 3.4, 3.5, 4.1, 4.4

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