Bayesian Games

Game Theory · Module 4 — Information And Beliefs

Node 4.1

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Information and Beliefs **Node:** 4.1

A Bayesian game (Harsanyi, 1967–68) is a game of **incomplete information**: players have private information about their preferences, their capabilities, or the state of the world. Harsanyi's insight was that any such situation can be modeled as a game in which nature draws a **type** for each player at the start, each player privately observes their own type, and then players act. This "Harsanyi transformation" converts incomplete information into imperfect information within a larger game tree, which can then be analyzed using the standard tools of Module 1–3. The node introduces the formal apparatus — types, type spaces, common priors — and sets up Bayesian Nash equilibrium (4.3) as the solution concept.

Formal Definition

A **Bayesian game** is a tuple

$$G^B = (N, \{\Theta_i\}, \{S_i\}, p, \{u_i\}),$$

where:

- N is the finite set of players.
- Θ_i is the set of possible **types** for player i. A type $\theta_i \in \Theta_i$ is a complete specification of player i's private information (preferences, beliefs, capabilities). The type profile is $\theta = (\theta_1, \dots, \theta_n) \in \Theta = \prod_i \Theta_i$.
- S_i is the set of actions available to player i (assumed independent of type for notational simplicity; can be generalized).
- $p \in \Delta(\Theta)$ is the **common prior** distribution over type profiles. It is common knowledge among players.
- $u_i : S \times \Theta \to \mathbb{R}$ is player i's payoff, which depends on the full action profile and the full type profile.

Type-contingent strategy. A pure strategy for player i is a function $s_i : \Theta_i \to S_i$: a plan of what action to take for each possible realization of own type. Mixed strategies are probability distributions over such functions, or equivalently type-contingent behavioral strategies.

Interim belief. Given own type θ_i, player i's belief about others' types is the conditional distribution $p(\theta_{-i} \mid \theta_i)$, derived from the common prior via Bayes' rule:

$$p(\theta_{-i} \mid \theta_i) = \frac{p(\theta_i, \theta_{-i})}{\sum_{\theta'_{-i}} p(\theta_i, \theta'_{-i})}.$$

Interim expected payoff. Given own type θ_i and a type-contingent strategy profile s, player i's expected payoff is

$$U_i(s \mid \theta_i) = \sum_{\theta_{-i}} p(\theta_{-i} \mid \theta_i) \cdot u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}).$$

The **Harsanyi transformation** represents this as a game of imperfect information: a chance node at the root draws $\theta \sim p$, each player observes only θ_i, then players choose actions.

Intuition

Incomplete information is the rule, not the exception. In any realistic strategic situation, players know things about themselves that others don't — their valuations in an auction, their costs in a market, their private signals about the state of the world.

Harsanyi's move was to *absorb this incompleteness into the game specification.* Instead of saying "player 1 doesn't know player 2's type," we say "there is a chance move at the start that draws types, and player 1 observes only their own type." This reduces incomplete information to imperfect information — a tree with chance nodes and information sets — and the standard tools apply.

The common prior is a strong assumption: it says all players agree on the distribution of types before observing their own type. They may update this to different posterior beliefs after observing their types, but the starting point is common knowledge. Under the common prior, differences in beliefs arise only from differences in information, not from differences in priors.

A type can encode any private information: preferences (how much you value a good), capabilities (your cost function), or beliefs about the state of the world (what you think nature has done). Harsanyi showed that all of these reduce to a type space, provided you're willing to make the type space big enough.

Structural Role

Bayesian games are the ontological foundation for every subsequent incomplete-information topic:

- **4.1** $\Rightarrow$ **4.2.** Types and type spaces are formalized further; hierarchies of beliefs.
- **4.1** $\Rightarrow$ **4.3.** Bayesian Nash equilibrium is the solution concept.
- **4.1** $\Rightarrow$ **4.4.** Perfect Bayesian equilibrium extends Bayesian Nash to sequential games.
- **4.1** $\Rightarrow$ **4.5.** Signaling games are two-stage Bayesian games.
- **4.1** $\Rightarrow$ **6.1.** Mechanism design is the inverse problem: design the game so Bayesian Nash equilibria yield desired outcomes.
- **4.1** $\Rightarrow$ **6.3.** Auction theory uses Bayesian games to model private valuations.
- **4.1** $\Rightarrow$ **4.6.** Sequential equilibrium is the belief-consistent refinement.

The introduction of types is also the formal entry point for poker game theory: each player's hole cards are their type, the common prior is the uniform deal, and the resulting Bayesian game is the full poker game.

Mathematical Development

The Harsanyi transformation

Start with incomplete information: player i has payoff function $u_i(\cdot, \theta_i)$ that depends on a private parameter $\theta_i \in \Theta_i$. Player j does not know θ_i. Instead of "player j has no idea," assume player j has a *probabilistic belief* $p_j(\theta_i)$.

Under a common prior, all players share the same joint distribution $p(\theta)$, with $p_j(\theta_i)$ the appropriate marginal. The Bayesian game is then:

1. Chance draws $\theta \sim p$ at the root.
2. Player i observes θ_i (and nothing else).
3. Players choose actions simultaneously (or sequentially, for extensive-form Bayesian games).
4. Payoffs are $u_i(s, \theta)$ at the leaves.

This is a game of imperfect information (the information sets group nodes differing in unseen θ_{-i}). The extensive-form machinery of Module 3 applies.

Type independence and correlated types

Independent types. $p(\theta) = \prod_i p_i(\theta_i)$. Each player's type is drawn independently. This is the simplest case and is standard in auction theory.

Correlated types. $p(\theta)$ does not factor. Correlation represents shared information, e.g., common-value auctions where all bidders have noisy signals about the same underlying value.

Conditional independence. Types are independent given some "state" variable, but marginal types are correlated.

Common priors and no-speculative-trade

Aumann's agreement theorem (1976). Under a common prior and common knowledge of the posterior, two Bayesian agents cannot "agree to disagree" about the expected value of any random variable. More strongly, they must have the same posterior.

No-speculative-trade theorem (Milgrom–Stokey, 1982). In a market with common priors, no trades can occur solely on the basis of private information — because every trade requires a buyer and seller whose beliefs diverge from the common prior, which Aumann's theorem rules out.

These results highlight how strong the common-prior assumption is. Dropping it gives a different theory (with "belief heterogeneity").

The type space as belief hierarchy

A rich type for player i includes not just their own payoff parameter but also their belief about others' parameters, their belief about others' beliefs about their parameter, and so on — an infinite hierarchy. Harsanyi showed that this hierarchy can always be encoded as a point in some finite-dimensional type space (the "universal type space"), provided beliefs are coherent. Mertens–Zamir (1985) formalized this and proved existence of a universal type space.

For most applications, one uses a finite type space with types chosen to capture the relevant private information directly (e.g., Θ_i = set of hole-card combinations in poker).

Worked Example

A simple Bayesian coordination game

Two players choose $\{A, B\}$. Player 1's payoff from coordination is known to be 2; Player 2's payoff from coordination depends on Player 2's type. Player 2 has two types, $\theta_2 \in \{H, L\}$, with $p(H) = 0.4, p(L) = 0.6$.

Player 2's payoff: - Type H: $u_2(A, A) = 3, u_2(B, B) = 1, u_2(\text{mismatch}) = 0$. - Type L: $u_2(A, A) = 1, u_2(B, B) = 3, u_2(\text{mismatch}) = 0$.

Player 1's payoff: $u_1(A, A) = 2, u_1(B, B) = 2, u_1(\text{mismatch}) = 0$. (Independent of Player 2's type.)

Player 1's interim belief. $p(H \mid \theta_1) = 0.4$ since θ_1 is empty (no private info) — the prior itself.

Player 1's expected payoff from A (supposing Player 2 plays best response given their type): Player 2 type H plays A (prefers matching A = 3 to mismatching = 0); type L plays B (prefers matching B = 3). So Player 1's expected payoff from A is $0.4 \cdot 2 + 0.6 \cdot 0 = 0.8$. From B: $0.4 \cdot 0 + 0.6 \cdot 2 = 1.2$. Player 1 plays B.

Bayesian Nash equilibrium: Player 1 plays B, Player 2 plays $s_2(H) = A, s_2(L) = B$. Verify: given Player 1 plays B, Player 2 type H gets 0 from A and 1 from B — type H should switch to B. Re-iterate.

Corrected analysis: given Player 1 plays B, type H best response is B (payoff $1 > 0$), type L best response is B (payoff $3 > 0$). So Player 2's strategy is $s_2 = (B, B)$. Player 1's best response to both types playing B: expected payoff from B is 2, from A is 0 — play B. Confirmed equilibrium: $(B, (B, B))$, payoffs $(2, \text{type-weighted})$.

Auction as a Bayesian game

First-price sealed-bid auction with two bidders. Each bidder has a private valuation $v_i \sim U[0, 1]$, drawn independently. Bidders submit bids simultaneously; higher bid wins and pays their bid.

- Types: $\Theta_i = [0, 1]$, $p(v) = \text{Uniform}[0, 1]^2$.
- Strategy: $s_i : [0, 1] \to [0, \infty)$, the bid as a function of valuation.
- Payoff: $u_i(b_1, b_2, v_i) = (v_i - b_i) \cdot \mathbb{1}[b_i > b_{-i}]$.

Bayesian Nash equilibrium (derived in Node 6.3): symmetric bidding with $s_i(v_i) = v_i/2$ for two bidders (and $v_i \cdot (n - 1)/n$ for n bidders). Each bidder shades their bid below their valuation because winning with a higher bid yields lower surplus.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Type θ_i	Your hole cards — the private information you hold
Type space Θ_i	$\binom{52}{2} = 1326$ possible 2-card combinations
Common prior $p(\theta)$	Uniform over all valid joint dealings of hole cards (no two players holding the same card)
Interim belief $p(\theta_{-i} \mid \theta_i)$	Your "range-the-opponent" reasoning: given you hold AK, what are the possible hands they have?
Type-contingent strategy $s_i(\theta_i)$	Your range strategy: a plan specifying what to do with each possible hole-card combination
Common prior assumption	Everyone at the table agrees the deck is uniformly shuffled

The "type" is literally your hole cards. When poker players talk about "playing your range" or "thinking in ranges," they are thinking about the Bayesian game structure: each player has a private type drawn from a type space, plays a type-contingent strategy, and the opponents do not know the realization. Harsanyi's transformation is what makes this a well-defined game theory problem.

Concrete situation

Flop decision as a Bayesian decision problem. You are BTN, facing a flop check from BB on $Kd9s4c$.

- **Your type:** your specific hole cards, say $A\clubsuit K\clubsuit$.
- **Opponent's type space:** all hole card combinations consistent with BB's preflop calling range.
- **Common prior on opponent's type (given their preflop action):** the distribution over combinations in BB's call-preflop range.

- **Interim belief:** your posterior over BB's hole cards, updated for the fact that they checked the flop (which for typical solver ranges includes a fairly wide check-call range).
- **Your strategy:** a type-contingent rule — "with AKcc, bet 33%; with 72o, give up / check" — a specific function from your hole-card type to an action.

Bayesian Nash structure. At equilibrium, your strategy is a best response to BB's strategy given your interim beliefs, and vice versa. The joint fixed point is the Bayesian Nash equilibrium of the poker game (restricted to the subgame rooted at this flop spot with these input ranges). The solver computes this fixed point.

What this understanding buys you

"Thinking in ranges" is Bayesian game-theoretic thinking. When a coach tells you "don't think about their hand, think about their range," they are translating the Bayesian game framework into poker jargon. Your opponent's type is private; the strategic object of interest is the *distribution* over their types, and you respond to the distribution, not to a specific realization.

Bayesian updating is the core move. Every opponent action updates your posterior over their type. Preflop, you have a prior (random 2 cards or filtered by position). They raise — update. They call your 3-bet — update. They check-call the flop — update. Each action narrows their range by Bayesian conditioning, which is exactly $p(\theta_{-i} \mid \text{history})$ computed via Bayes' rule.

You are also a type from their perspective. The symmetry is crucial: you are thinking about their range, but they are simultaneously thinking about yours. In equilibrium, each of you is best-responding to the other's type-contingent strategy, and the joint profile is a fixed point. Both ranges are common knowledge in the common-prior sense; what's hidden is the type realizations.

Common prior is a real assumption. If you and opponent disagree about "what the preflop range is," you are not in a common-prior Bayesian game. This is a form of belief heterogeneity and can create exploitable gaps. Most solver study is based on a shared assumption about what ranges "should" look like in equilibrium — violating this (by playing non-standard ranges) is a form of exploitation.

Cross-Links

- **1.1 Players, strategies, payoffs** — the base ontology.
- **3.2 Information sets** — the extensive-form representation of types.
- **4.2 Types and type spaces** — formalization of private information.
- **4.3 Bayesian Nash equilibrium** — the solution concept.
- **4.4 Perfect Bayesian equilibrium** — sequential refinement.
- **4.5 Signaling games** — two-stage Bayesian games.
- **6.1, 6.3 Mechanism design, auctions** — applied Bayesian games.
- **Statistics — Bayesian inference** — the mathematical toolkit.

Structural Compression

Invariant: A Bayesian game is a normal-form game enriched with a type space for each player and a common prior over type profiles, representing incomplete information as a lottery over strategic situations; the Harsanyi transformation converts any such situation into a standard imperfect-information extensive-form game.

Mechanism: Type chance node at the root, private type observation by each player, type-contingent strategies, and interim Bayesian belief computations over opponents' types.

Cross-System Echo: Bayesian games are the game-theoretic instantiation of the general pattern "uncertain parameters observed privately by agents": hidden-variable models in statistics, partial-observation settings in reinforcement learning, and private-state channels in information theory all instantiate the same information asymmetry with an appropriate prior.

Exercises

1. Write down the Bayesian game structure for a first-price sealed-bid auction with two bidders and private uniform valuations.
2. Compute the interim belief of player i of type θ_i about θ_{-i} in a Bayesian game with a given common prior.
3. Consider a coordination game where Player 2 has two equally likely types, each with different preferred actions. Find the Bayesian Nash equilibrium.
4. Show that under a common prior, two Bayesian agents with the same information must have the same posterior (Aumann agreement).
5. Construct a Bayesian game with correlated types and write the conditional distribution $p(\theta_{-i} \mid \theta_i)$.
6. Formulate 2-card heads-up hold'em with uniform deal as a Bayesian game. Specify Θ_i, p, and the payoff function at showdown.
7. **Argue, structurally,** why the Harsanyi transformation is a "free" reduction: every game of incomplete information can be represented as a game of imperfect information with a chance move, without loss of strategic content. Identify what additional assumption — beyond the type space itself — is required to make the transformation work.

Game Theory · Module 4 — Information And Beliefs · Node 4.1
Concepts: bayesian-nash-equilibrium, conditional-probability, bayesian-updating, prior-posterior, incomplete-information
Prerequisites: 1.1, 3.2
Enables: 4.2, 4.3, 4.4, 4.5, 6.1, 6.3

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