Graph Games and Strategic Topology

Game Theory · Module 8 — Network Games

Node 8.1

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Network Games **Node:** 8.1

Module 8 is the point at which game theory stops being a theory of "n players in a room" and becomes a theory of **interaction topologies**. In Modules 1–7, when we wrote $u_i(s_i, s_{-i})$, the index $-i$ meant "everyone else," and the combinatorial object holding the game together was the complete Cartesian product $S_1 \times \cdots \times S_n$. Every player's payoff could, in principle, depend on every other player's action. That is the *complete-graph* assumption: the interaction structure is K_n, and it is typically left implicit. Once we take interaction topology seriously — once we recognize that Alice only interacts with her neighbors, that influence propagates along edges rather than through a room — we are in the world of **graph games**. This node establishes the framework: a game on a graph, players on nodes, payoffs depending only on the actions of graph-neighbors, and the structural consequences of this locality. Every subsequent node in Module 8 — coordination (8.2), diffusion (8.3), public goods (8.4), formation (8.5), complementarities (8.6) — is a specialization of the general graph-game framework introduced here.

Formal Definition

A **graph game** (or **network game**) is a tuple

$$\Gamma = (N, G, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$$

where:

- $N = \{1, \ldots, n\}$ is the set of **players**, each identified with a node.
- $G = (N, E)$ is a graph (directed or undirected) on N, called the **interaction network**. For $i \in N$, write $\mathcal{N}_i = \{j : (i,j) \in E\}$ for the **neighborhood** of i, and $d_i = |\mathcal{N}_i|$ for the **degree** of i.
- S_i is player i's strategy set.
- $u_i : S_i \times S_{\mathcal{N}_i} \to \mathbb{R}$ is player i's payoff function, **depending only on i's own strategy and the strategies of i's neighbors**.

The **locality axiom** is the defining constraint: u_i factors through the neighborhood, $u_i(s) = \tilde{u}_i(s_i, s_{\mathcal{N}_i})$. A game where every u_i depends on all of s_{-i} is the special case $G = K_n$ (complete graph), recovering the standard normal-form game.

A **symmetric** graph game is one where u_i has the same functional form for every i; payoff differences come entirely from differences in $s_{\mathcal{N}_i}$ (which depend on the graph position). A **linear-aggregative** graph game is one where $u_i(s_i, s_{\mathcal{N}_i}) = f(s_i, \sum_{j \in \mathcal{N}_i} s_j)$ — payoff depends on own action and the *sum* of neighbor actions. Linear-aggregative games are the workhorse of the network-games literature because they admit closed-form equilibria via the graph Laplacian.

A **Nash equilibrium** of a graph game is a profile s^* such that for each i,

$$s_i^* \in \arg\max_{s_i \in S_i} \tilde{u}_i(s_i, s^*_{\mathcal{N}_i}).$$

The locality axiom means the best-response computation is local: player i only needs to know the actions of neighbors, not of distant players. This local-best-response structure is what makes network games computationally and conceptually tractable.

Intuition

The move from "complete interaction" to "graph interaction" changes the game in three ways. First, **payoffs are local**: each player's welfare depends only on direct connections. Second, **information can be local**: best responses can be computed by looking only at neighbors. Third, and most important, **equilibrium structure inherits graph structure**: the spectrum of the interaction graph (eigenvalues, Bonacich centrality, spectral radius) appears in closed-form equilibrium expressions. A dense interaction graph and a sparse interaction graph produce qualitatively different strategic landscapes, and the graph itself becomes a parameter of the theory.

Think of the difference between "everyone in the room votes" and "everyone talks to their neighbors, and behavior propagates." The first is a normal-form game on K_n; the second is a graph game on a sparse network. In the first, no topology matters — the outcome depends only on payoffs and equilibrium selection. In the second, topology is everything: who is adjacent to whom determines who influences whom, which determines which equilibria are stable, which cascades can propagate, and whose individual position in the graph gives them strategic leverage.

The strategic role of **position**. In a graph game, two otherwise identical players can have radically different equilibrium payoffs because they sit at different positions in the graph. A player at the center of a star has payoffs averaged over many neighbors; a player at the tip of a spoke has a payoff determined by the center alone. Degree, centrality, spectral position, and cluster membership all become strategic assets. This is the game-theoretic foundation of "network power" — the idea that in a connected system, *where* you are matters as much as *what* you do.

A useful framing: a graph game is a factor-graph specification of a strategic situation. The graph tells you *which pieces of the joint strategy profile matter for which player's payoff.* Anything outside the neighborhood is irrelevant to that player's best response, which means best-response dynamics, learning dynamics, and equilibrium computation all decompose along the graph. The locality axiom is what makes network games fit into Module 8 — it is simultaneously the defining restriction and the source of all the tractability gains.

Structural Role

8.1 is the foundational node of Module 8. Every subsequent node in the module is a specialization of the graph-game framework to a particular class of games:

- **8.2 Coordination on networks** specializes to games where $u_i(s_i, s_{\mathcal{N}_i})$ rewards matching with neighbors — producing Morris-style contagion, thresholds, and best-response dynamics on graphs.
- **8.3 Diffusion and cascades** specializes to games where adoption decisions depend on the fraction of adopted neighbors — producing the Kempe–Kleinberg–Tardos framework of influence maximization.
- **8.4 Public goods on graphs** specializes to games where effort is a local public good — producing the Bramoullé–Kranton specialization theory.
- **8.5 Network formation games** lets the graph G itself be a strategic variable, endogenizing the topology.
- **8.6 Strategic complementarities** restricts attention to games where higher actions by neighbors make higher own actions more attractive — producing Topkis's supermodularity theory and the global-games refinement.

Beyond Module 8, the graph-game framework appears in Module 9 (learning on networks), Module 10 (mechanism design on networks), and throughout the strategic systems integration modules. It is also the point of contact with network science, statistical physics (Ising models on graphs are special cases of coordination graph games), and distributed optimization.

Mathematical Development

Linear-quadratic network games

The canonical tractable class is the **linear-quadratic graph game**: each player i chooses an effort level $x_i \in \mathbb{R}_{\geq 0}$ with payoff

$$u_i(x_i, x_{\mathcal{N}_i}) = \alpha_i x_i - \tfrac{1}{2}x_i^2 + \phi \sum_{j \in \mathcal{N}_i} x_i x_j.$$

Here α_i is an idiosyncratic marginal benefit, the $-\frac{1}{2}x_i^2$ term is a private cost, and ϕ is the **strategic interaction parameter** — positive for complementarities (neighbors' efforts make my effort more attractive), negative for substitutes (neighbors' efforts make my effort less attractive).

Best response. Setting $\partial u_i / \partial x_i = 0$:

$$x_i^* = \alpha_i + \phi \sum_{j \in \mathcal{N}_i} x_j^*.$$

In matrix form, let A be the adjacency matrix of G:

$$x^* = \alpha + \phi A x^* \iff (I - \phi A)x^* = \alpha.$$

If $|\phi| < 1/\rho(A)$, where $\rho(A)$ is the spectral radius of A, then $(I - \phi A)$ is invertible and

$$x^* = (I - \phi A)^{-1}\alpha = \sum_{k=0}^{\infty} \phi^k A^k \alpha.$$

This is the **Bonacich centrality** of α under the interaction graph G with decay parameter ϕ. The walk-counting interpretation: $(A^k)_{ij}$ counts walks of length k from i to j, so x_i^* aggregates influences from nodes at all graph distances, weighted by ϕ^k.

Theorem (Ballester–Calvó-Armengol–Zenou, 2006). The unique Nash equilibrium of the linear-quadratic network game is proportional to the Bonacich centrality vector of the interaction graph. Equilibrium effort at node i is monotone in i's Bonacich centrality.

This is the most celebrated result in network games: a player's equilibrium effort is determined by their **graph-theoretic centrality**, and the centrality measure that matters is not degree or betweenness but the walk-counting Bonacich measure indexed by ϕ.

Best-response dynamics on graphs

Starting from any profile, run asynchronous best-response updates: at each step, a player reads their neighbors' current actions and updates their own to a best response. In a linear-quadratic game with $|\phi| < 1/\rho(A)$, this dynamic is a contraction and converges globally to the unique equilibrium. In games with strategic complementarities (8.6), best-response dynamics converge monotonically to equilibria because the joint best-response map is monotone in the Topkis sense.

The **speed of convergence** depends on graph structure — specifically on the spectral gap of A. Well-connected graphs (high spectral gap) converge fast; graphs with bottlenecks or communities converge slowly. This is the same spectral-gap phenomenon that governs mixing times of random walks, and the analogy is not accidental: best-response dynamics on a linear-quadratic game *are* a linear dynamical system driven by A.

Local vs global equilibria

Because payoffs depend only on neighbors, a graph game can have equilibria that are locally consistent (every player is best-responding to their neighbors) but globally inefficient. The gap between the Nash equilibrium and the social optimum depends on graph structure — cut-vertices, clustering, and the presence of long paths all matter. This gap is the origin of the "specialization" result of Bramoullé–Kranton (8.4) and the "stability-efficiency tradeoff" of Jackson–Wolinsky (8.5).

Worked Example

Linear-quadratic game on a star

Consider a star graph $K_{1,n}$ with one central node c connected to n peripheral nodes $p_1, \ldots, p_n$. Each player has the same idiosyncratic marginal benefit $\alpha_i = \alpha$. Strategic parameter $\phi > 0$, small enough for uniqueness.

Best responses:

$$x_c^* = \alpha + \phi \sum_{j=1}^{n} x_{p_j}^*, \qquad x_{p_j}^* = \alpha + \phi x_c^*.$$

By symmetry, all peripherals choose the same value x_p^*. Substituting:

$$x_c^* = \alpha + n\phi x_p^*, \qquad x_p^* = \alpha + \phi x_c^*.$$

Solving:

$$x_c^* = \frac{\alpha(1 + n\phi)}{1 - n\phi^2}, \qquad x_p^* = \frac{\alpha(1 + \phi)}{1 - n\phi^2}.$$

For $\phi > 0$, $x_c^* > x_p^*$: the central node exerts higher effort. The ratio $x_c^*/x_p^* = (1 + n\phi)/(1 + \phi)$ grows with n. A player at a more central position chooses a larger action because *more* neighbors' actions feed back to it, amplifying the private return on effort.

Structural reading. The star exhibits a pure centrality premium. Two players with identical preferences choose different actions because they sit at different graph positions. The center's effort is a weighted average of idiosyncratic benefit α and the feedback from n peripherals; the peripherals' effort feels only the center. This is the kernel of the Bonacich-centrality result: in any graph, equilibrium effort tracks walk-weighted centrality.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Nodes N	Players in a community, coaches, content creators, study groups
Interaction graph G	Who shares hands with whom, whose Discord you are in, who you talk strategy with
Neighborhood $\mathcal{N}_i$	Your immediate circle of exchange partners
Strategic action x_i	Time invested in solver study, range construction, review
Complementarity $\phi > 0$	Study is more valuable when peers study (shared language, peer review)
Bonacich centrality	"Plugged-in-ness": reach into the aggregate study network

The poker ecosystem is **not** a complete graph. It is a sparse network of regulars, study groups, Discords, coaches, staked stables, and content creators. Your information and strategy circulate only through the edges you actually have. Two equally talented players who sit at different positions in this graph will end up with different strategies and different win rates, not because of talent but because of topology.

Concrete situation

Consider a mid-stakes NLHE regular, Alex, who is part of a 5-person study group plus two Discord servers. Alex's "interaction graph" includes the other 4 study-group members (dense edges — daily hand discussion), roughly 20 active Discord contacts (sparse edges — occasional spot review), and a larger tail of 200+ lurkers who read but do not contribute. Alex's neighborhood in the graph game is the dense core: 4 study-group members plus maybe 5 high-touch Discord contacts.

Alex's effort level (hours on solver work) is a best response to the study network's effort. If the group is running deep on a particular topic (turn probe-bet frequencies in single-raised pots), Alex's marginal benefit from doing the same is high — they can share findings, compare trees, catch each other's errors. If the group is checked out, Alex's marginal benefit collapses: why solve in isolation when the work will not circulate? This is a **positive-ϕ** linear-quadratic game on Alex's study graph, and Alex's equilibrium effort is proportional to the Bonacich centrality of Alex within that graph.

Now contrast with Blake, a similarly skilled regular who is in no study group and two low-activity Discords. Blake's neighborhood is effectively empty — isolated node. Blake's best-response effort is the base level α with no multiplier. Alex ends up studying much more than Blake even if they start with identical α. And critically, Alex's study time is determined by Alex's position in the graph, not by innate traits.

What this understanding buys you

Your poker "work ethic" is not fully yours. A significant fraction of your study and game effort is a best response to your local network's effort. If you are in a strong study group, you will naturally put in more work. If you are isolated, you will naturally put in less. This is not a moral failing — it is a structural equilibrium of a graph game with strategic complementarities. The right intervention is not to "try harder" but to **change your graph position**: join a study group, cultivate a small set of high-activity contacts, become central in a live network of regulars who care. Changing your neighborhood changes your ϕA multiplier, which changes your equilibrium effort.

Centrality has a dual: exposure. A very central position in the poker information network also means that your strategy is visible to many people. If you are the center of a 20-person Discord, your hand histories circulate faster than a peripheral player's, and anything you do gets seen, studied, counter-exploited. The same centrality that raises your effort also raises your visibility. The optimal graph position trades off effort-amplification against information-leakage, and 8.1 is the framework in which to reason about that tradeoff.

Cross-Links

- **1.1 Players, strategies, payoffs** — graph games are the locality-constrained specialization.
- **1.4 Best response** — extends to the local best response in graph games.
- **2.1 Nash equilibrium** — specializes to equilibria of linear-quadratic graph games via Bonacich centrality.
- **8.2 Coordination on networks** — the next specialization, to coordination payoffs.
- **8.4 Public goods on graphs** — Bramoullé–Kranton substitutes game.
- **8.5 Network formation games** — endogenizing the graph itself.
- **8.6 Strategic complementarities** — the monotone-comparative-statics specialization.
- **9.3 Learning on graphs** — best-response dynamics and convergence via spectral gap.
- **Network science** — the graph-theoretic layer on which everything in Module 8 is built.

Structural Compression

Invariant: A graph game is a strategic situation in which each player's payoff depends only on their own action and the actions of graph-neighbors, and equilibrium structure inherits the spectral and topological properties of the interaction graph.

Mechanism: Locality axiom $u_i = \tilde{u}_i(s_i, s_{\mathcal{N}_i})$; best-response updates depending only on neighbors; equilibrium of linear-quadratic games given by $x^* = (I - \phi A)^{-1}\alpha$, i.e. Bonacich centrality; spectral radius governs existence and uniqueness.

Cross-System Echo: Graph games are the game-theoretic analogue of **local-interaction models in statistical physics** (Ising models, spin glasses) and **factor graphs in probabilistic inference** (belief propagation, Markov random fields). In all three, a global object (joint distribution, joint strategy profile, joint energy) factorizes along a sparse graph, and global structure is controlled by spectral properties of that graph. The convergence of best-response dynamics is structurally identical to the convergence of belief propagation and Glauber dynamics — same spectral-gap bounds, same local-to-global logic.

Exercises

1. For the linear-quadratic game on a cycle C_n with uniform $\alpha_i = \alpha$ and parameter ϕ, compute the symmetric Nash equilibrium effort level as a function of ϕ and n. Compare to the complete-graph and star-graph cases.
2. Show that in a linear-quadratic graph game with $\phi > 0$, the equilibrium effort x_i^* is monotone non-decreasing in d_i holding other neighbors fixed. Is the same true in terms of Bonacich centrality across graphs of different structure?
3. A star $K_{1,n}$ has spectral radius $\rho = \sqrt{n}$. Derive the uniqueness condition $|\phi| < 1/\rho$ for the linear-quadratic game on the star and interpret the failure mode when ϕ exceeds this bound.
4. Construct a graph game on 4 nodes (two disjoint edges) where best-response dynamics converges to one of two asymmetric equilibria depending on initial condition. Identify the structural source of multiplicity.
5. Write down the linear-quadratic game on a graph with $\phi < 0$ (substitutes). Show that effort is *anti-monotone* in neighbors' effort and that equilibria can be non-symmetric even on symmetric graphs. Relate to Bramoullé–Kranton specialization (preview of 8.4).
6. A graph game has the structure: nodes are players, edges indicate "share strategy files," and the strategic action is "depth of solver study (hours)" with complementarities on study partners. Formalize this as a linear-quadratic game, specify parameters, and predict which players will study the most given degree-based graph statistics.
7. **Argue, structurally,** why the graph-game framework is strictly richer than the complete-graph normal-form game, and identify what strategic phenomena become visible in graph games that are invisible in the complete-graph case. (Hint: think about centrality, specialization, local multiplicity, and the role of graph position.)

Game Theory · Module 8 — Network Games · Node 8.1
Concepts: graph-theory, adjacency-matrix, nash-equilibrium
Prerequisites: 1.1, 1.4, 2.1
Enables: 8.2, 8.3, 8.4, 8.5, 8.6