Coordination on Networks

Game Theory · Module 8 — Network Games

Node 8.2

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Network Games **Node:** 8.2

A **coordination game** rewards matching: if my neighbor plays A, I want to play A; if they play B, I want to play B. On a two-person complete graph this is just the Stag Hunt or its cousins, with two pure equilibria (A, A) and (B, B) and the standard equilibrium-selection question. When we put the same coordination game onto a sparse graph, something qualitatively new happens: players in different regions of the graph can be **locally** coordinated on different conventions, and the question becomes whether one convention can **spread** to the other. The classical analysis is due to Stephen Morris (2000), "Contagion": on an infinite graph with a threshold coordination game, when can a single deviation trigger contagion to the whole population? The answer depends on the graph's **cohesion structure** and on the threshold at which players switch. This node develops the Morris framework, the critical threshold for contagion, best-response dynamics on coordination graphs, and the conceptual bridge to the diffusion models of 8.3.

Formal Definition

A **coordination graph game** is a graph game $(N, G, \{S_i\}, \{u_i\})$ where every player has the same binary strategy set $S_i = \{A, B\}$ and a payoff of the form

$$u_i(s_i, s_{\mathcal{N}_i}) = \sum_{j \in \mathcal{N}_i} \pi(s_i, s_j),$$

where π is a symmetric 2-by-2 coordination payoff matrix with

$$\pi(A, A) > \pi(B, A), \qquad \pi(B, B) > \pi(A, B).$$

Both diagonals dominate the off-diagonals, so coordination with each neighbor is locally optimal. The prototypical normalization:

	A	B
A	(a, a)	$(0, 0)$
B	$(0, 0)$	(b, b)

with $a, b > 0$. Player i's payoff from playing A when a fraction q_i of neighbors play A is $a \cdot d_i q_i$; from playing B is $b \cdot d_i(1 - q_i)$. Best-response condition:

$$\text{play } A \iff a q_i \geq b(1 - q_i) \iff q_i \geq q^* := \frac{b}{a + b}.$$

The threshold $q^* \in (0, 1)$ is the **critical fraction** of A-playing neighbors at which player i is indifferent.

Contagion threshold (Morris). An action A is said to be **contagious** on graph G with threshold q^* if starting from a finite set of A-playing seeds with the rest playing B, the best-response dynamics eventually spread A to the entire graph. The **contagion threshold** $q^*(G)$ is the largest q^* for which A is contagious.

Morris shows that on suitably regular infinite graphs, $q^*(G)$ is a well-defined number in $(0, 1/2]$, and it is determined by a **cohesion** property of the graph.

Cohesion. A subset $S \subseteq N$ is **p-cohesive** if every node in S has at least a fraction p of its neighbors inside S. Morris's key theorem: A is contagious from a finite seed iff the complement of the seed contains no infinite $(1 - q^*)$-cohesive set. Equivalently: contagion happens iff there is no "cohesive region" large enough to resist conversion.

Intuition

The two-player story of a coordination game is equilibrium selection: both (A, A) and (B, B) are Nash, and we argue on theoretical grounds (risk dominance, payoff dominance, cheap talk, focal points) about which will be played. The network story is something else. Imagine a large population playing the coordination game, arranged on a sparse graph, each playing a best response to their neighbors. Equilibria include the all-A and all-B profiles, but also **mixed equilibria** where some regions of the graph play A and other regions play B, with the boundary between them kept stable because boundary players have enough same-action neighbors to justify their choice.

Morris asks: starting from the all-B equilibrium, if a small group of seed players switches to A, will the switch **spread**? Not if A is unattractive (high threshold). Not if the rest of the graph is tightly knit around B. The question becomes: is there any graph region that is tightly enough coordinated on B to be immune to the spreading A?

The concept of **cohesion** captures this immunity. A region is cohesive if its nodes have most of their neighbors inside the region. A cohesive B-region is self-sustaining: its border nodes see only a small fraction of outside A-neighbors and so fall below the threshold for switching. The tightness of this cohesion is exactly what q^* measures: a lower q^* (A is more attractive) requires less cohesion to resist; a higher q^* requires more.

The critical-threshold picture ties it together. Every graph has some maximum q^* below which a finite seed can propagate A to the whole graph and above which it cannot. This number is the **contagion threshold** of the graph. On the infinite line $\mathbb{Z}$, the contagion threshold is $1/2$: any action with $q^* < 1/2$ can spread from a single seed. On a 4-regular lattice, it is lower — locally cohesive regions can resist. On random graphs with high clustering, it is lower still.

The deepest insight is that **coordination on a network is a game of cohesion vs reach**. A tightly clustered community can sustain local conventions indefinitely; a well-mixed network is vulnerable to whichever convention is more attractive. This is the game-theoretic foundation of much of social-norms research, language adoption, technology standards, and political polarization.

Structural Role

8.2 sits between 8.1 (general graph games) and 8.3 (diffusion and cascades). It specializes the graph-game framework to coordination payoffs and develops the analytic tools — thresholds, cohesion, contagion bounds — that the diffusion node will generalize.

- **From 8.1:** inherits the graph-game framework and the local-best-response structure.
- **To 8.3:** the Morris contagion model is a special case of the threshold-diffusion model of Kempe–Kleinberg–Tardos, which allows heterogeneous thresholds and richer payoff structures.
- **To 8.6:** coordination games are supermodular (strategic complementarities), so the results here are concrete instances of the general Topkis theory.
- **To Module 9:** best-response dynamics on coordination graphs is a particular case of learning in games, and convergence results here feed back into learning theory.

Mathematical Development

Best-response dynamics on a coordination graph

Consider the discrete-time asynchronous dynamic: at each step, a single player wakes up, reads the current neighbor actions, and switches to the best response. Equivalently, in the synchronous version, all players simultaneously update.

Define $n_A^t(i)$ as the number of A-playing neighbors of i at time t, and $q_i^t = n_A^t(i)/d_i$. Player i's best response is

$$s_i^{t+1} = \begin{cases} A & \text{if } q_i^t \geq q^* \\ B & \text{if } q_i^t < q^* \end{cases}$$

(ties broken arbitrarily). This is a **monotone** dynamic in the following sense: if A is a **complement** (q_i going up makes A more attractive), then the update rule is monotone in the partial order where "more A's" is higher. Consequently, starting from any profile, best-response dynamics converges to an equilibrium (no cycles), and starting from the all-B profile with a seed set S, the dynamics converges to the smallest equilibrium above "A on S, B elsewhere."

Morris's contagion theorem

Theorem (Morris, 2000). Let G be an infinite, bounded-degree graph and let $q^* \in (0, 1)$. A is contagious from some finite seed on G iff there is no infinite subset $T \subseteq N$ that is $(1 - q^*)$-cohesive (every node in T has at least a $1 - q^*$ fraction of its neighbors inside T).

The contagion threshold of G is

$$q^*(G) = \sup\{q^* : \text{no infinite } (1 - q^*)\text{-cohesive set exists}\}.$$

For any finite q^* strictly below $q^*(G)$, a finite seed can convert the whole graph to A. For any q^* strictly above $q^*(G)$, some cohesive B-region always survives.

Theorem. $q^*(G) \leq 1/2$ for every graph G.

Proof sketch: at the boundary between two regions (one all-A, one all-B), the boundary nodes have roughly half their neighbors on each side if the graph is locally homogeneous. So $q^* \leq 1/2$ is necessary for contagion in almost any structured graph.

The contagion threshold on specific graphs

Infinite line $\mathbb{Z}$: $q^*(\mathbb{Z}) = 1/2$. Every node has 2 neighbors; a seed of one A-node flips a neighbor if $1/2 \geq q^*$. Since strict inequality is needed for a single seed to start a chain, the dynamic actually flips when $q^* < 1/2$.

Infinite 2D lattice $\mathbb{Z}^2$: $q^*(\mathbb{Z}^2) = 1/2$ (Morris). A half-plane of A has boundary nodes with $q = 1/2$, so strict threshold $q^* < 1/2$ spreads.

Regular tree of degree d: $q^*(T_d) = 1/d$. Each node has d neighbors; a single A-neighbor is a fraction $1/d$. So any $q^* < 1/d$ spreads.

Random graphs with clustering: contagion thresholds are generally lower than for non-clustered graphs of the same mean degree, because clusters form cohesive B-regions that resist conversion.

Risk dominance vs contagion

In a symmetric 2×2 coordination game, the **risk-dominant** equilibrium is the one with higher q^* compared to its alternative. If $\pi(A, A) = a, \pi(B, B) = b$ with $a > b$, then (A, A) is **payoff-dominant** but its threshold for switching to it is $q^* = b/(a + b) < 1/2$, while (B, B) has threshold $1 - q^* > 1/2$ — (B, B) is harder to break out of, which makes B the **risk-dominant** strategy. Morris's contagion theorem says that **the risk-dominant equilibrium is the one that spreads**: the action that spreads from a finite seed is the one with lower threshold, i.e. the risk-dominant action. This is a beautiful alignment between risk dominance (a solution concept from two-player game theory) and contagion (a property of network dynamics).

Worked Example

Contagion on an infinite line

Let $G = \mathbb{Z}$ (infinite line). Coordination game with $a = 3, b = 2$, so $q^* = 2/5 = 0.4$.

Seed set $\{0\}$ starts playing A; everyone else plays B.

At time 1, node 0 is A, all others B. Node 1 has one A-neighbor and one B-neighbor, so $q_1 = 1/2 \geq 0.4$. Best response: A. Similarly node -1 switches to A.

At time 2, nodes $\{-1, 0, 1\}$ play A. Node 2 has one A-neighbor (node 1) and one B-neighbor (node 3), $q_2 = 1/2 \geq 0.4$. Switch to A. Similarly for node -2.

The wave of A spreads one node per time step on each side. Eventually the entire line converts.

Now raise the threshold: $a = 2, b = 3$, so $q^* = 3/5 = 0.6$. Node 1 has $q_1 = 1/2 < 0.6$, so does not switch. The seed $\{0\}$ is isolated at A and everyone else stays at B. The wave does not start.

Critical threshold on $\mathbb{Z}$: $q^*(\mathbb{Z}) = 1/2$. At exactly $q^* = 1/2$, the wave may or may not spread depending on tie-breaking; strict inequality in either direction gives a clean answer.

Contagion on a cohesive cluster

Consider a graph on 20 nodes: a tight cluster of 10 nodes forming a complete subgraph K_{10}, plus 10 additional nodes forming a line attached to one node of K_{10}.

Start with A seeded on the far tip of the line. With $q^* = 0.4$, A spreads along the line (each node has 2 neighbors, converting when one is A). It eventually reaches the attachment point to K_{10}.

Now ask whether A spreads into K_{10}. Each node in K_{10} has 9 neighbors inside the cluster and (for the attachment node) 1 outside. When the outside neighbor is A, the attachment node has $q = 1/10 = 0.1 < 0.4$. It does not switch. K_{10} is a (9/10)-cohesive set, which exceeds $1 - q^* = 0.6$, so it is immune.

Lesson: a tightly cohesive cluster is a firewall against contagion even for low thresholds. A well-connected community can preserve its convention indefinitely against invasions from a sparse periphery. This is the Morris theorem in action: the cluster's cohesion 9/10 exceeds $1 - q^* = 0.6$, so contagion is blocked.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Strategies A, B	Two competing "sizing trees" — e.g., 32% / 75% / overbet vs 50% / 100% / overbet
Payoff of coordinating with a neighbor	Easier to study and discuss hands when you and your study partners use the same size grid
Neighbor set	Your study group, coaching clients, content creators you follow
Threshold q^*	How many of your peers need to use the new sizing tree before you switch
Cohesive cluster	A study group that has converged internally on a convention
Contagion event	A new solver insight spreading through the poker ecosystem

Formal object	Poker instantiation
Risk dominance	The sizing tree that is easier to switch to

Concrete situation: spread of a sizing convention

Two years ago, most high-stakes NLHE regulars used a sizing tree with 75% pot as the default c-bet. A small group of solver-heavy regulars started using 33% pot (on certain board textures) because their solvers showed it was marginally higher EV in spots with low equity swings. This was a new "convention" — call it action A. Old convention is action B.

The payoff structure is a coordination game. If your study group uses 33%, you want to use 33% too — otherwise your hand discussions are apples-to-oranges, you cannot compare lines, and your solver runs do not map to the shared vocabulary. Within a tight study group, there are strong complementarities. Across the ecosystem, the graph is very sparse: you only coordinate with people you actually talk to.

Morris asks: does the 33% convention spread from the initial seeds to the whole ecosystem? The answer depends on two things: the **threshold** (how attractive is 33%?) and the **cohesion** of the existing 75% community. In the actual history, 33% did spread through the high-stakes online ecosystem (well-connected, low clustering, so low cohesion of any single subcommunity), but did not fully reach live cash games (tightly clustered local communities, high cohesion, resistance to change).

Concrete situation: Discord server language

A poker Discord server develops local conventions: certain abbreviations (SRP, 3BP, OOP-COOP), certain default sizes to reference in discussion, certain tags for hand history posts. These are pure coordination conventions — no inherent reason to prefer one over another, but if everyone uses them, communication is easier. New members face a coordination decision: do I use the server's conventions or my personal ones?

The game is locally cohesive: your "neighborhood" is the active posters, and the fraction of them using the server convention is near 1. The threshold for switching is low (small gain from matching), but the cohesion is overwhelming — you match. A new convention can only spread through the server if a cohesive subgroup adopts it first (the power-posters, the admins, the solver-study sub-channel) and then propagates outward.

What this understanding buys you

Convention change is a contagion problem, not a correctness problem. When you want to change how a group of players talks about or plays something, the question is not "is my idea correct?" but "what is the cohesion of the group's current convention, and does my idea beat the threshold?" You can be completely right about a new sizing tree or a new concept, but if the existing convention is cohesive enough, your idea will not spread — it will be trapped in a seed set and isolated. The intervention is to attack the cohesion: find and convert the **most-connected node** (a high-Bonacich-centrality regular or content creator), because their conversion breaks the cohesion of their local cluster and lowers q^* for their neighbors.

Your study group's language is a local convention. If you have been coordinating with a small group for a year, your shared vocabulary is a highly cohesive q^*-equilibrium. Joining a bigger community with a different convention feels like a cost — that is the coordination cost of switching, not a sign that the new community is wrong. And conversely, if you want your group to update to new ideas, you need to import a critical mass of converted nodes, not just make the argument once.

Online and live are different graphs. The online regular community is dense and low-clustering — contagion of new ideas is fast. The live cash game community is sparse and highly clustered — conventions persist for decades. This is not about intelligence or modernity; it is about the topology of the interaction graph.

Cross-Links

- **2.1 Nash equilibrium** — coordination games have multiple Nash; networks resolve selection.
- **8.1 Graph games** — coordination is the coordination-payoff specialization.
- **8.3 Diffusion and cascades** — generalizes Morris's contagion to richer threshold models.
- **8.6 Strategic complementarities** — coordination games are the canonical supermodular example.
- **4.6 Global games** — equilibrium-selection refinement that often picks the risk-dominant action, matching Morris's contagion result.
- **7.4 Evolutionary stability** — alternative selection concept giving the same risk-dominant answer in many cases.
- **Network science — epidemic models** — mathematical parallels to threshold contagion.

Structural Compression

Invariant: On a network coordination game with threshold q^*, an action is contagious from a finite seed iff no subset of the remaining nodes is sufficiently cohesive to resist conversion; the risk-dominant action is the one that spreads.

Mechanism: Binary best-response dynamics with threshold rule $s_i = A \iff q_i \geq q^*$; monotone dynamics converging to equilibrium; contagion blocked by $(1 - q^*)$-cohesive subsets.

Cross-System Echo: Morris contagion is the game-theoretic analogue of **bootstrap percolation** in statistical physics — a cellular automaton where cells activate when a fraction of neighbors are active. The same phase transitions, the same cohesion-based obstructions, the same lattice-vs-tree asymmetries appear in both. It is also parallel to the **voter model** of population genetics and to the **SIR threshold models** of epidemiology. The common structural feature: a local update rule depending on a fraction of neighbors in a particular state, propagating a wave whose reach is controlled by the graph's cohesion structure.

Exercises

1. Derive the threshold $q^* = b/(a + b)$ for the symmetric 2×2 coordination game with on-diagonal payoffs a and b and off-diagonal 0. Show that the risk-dominant action is the one with the smaller threshold.

2. Compute the contagion threshold of the infinite 2D lattice $\mathbb{Z}^2$ under 4-neighbor connectivity. Show that $q^*(\mathbb{Z}^2) = 1/2$ and identify what happens at exactly $q^* = 1/2$.

3. Show that on a regular tree T_d with $d \geq 3$, the contagion threshold is $1/d$. Compute the threshold for $d = 3, 4, 5$ and interpret why higher-degree trees are harder to convert.

4. Construct a finite graph with two cohesive communities of sizes 10 and 10 connected by a single edge. For a coordination game with $q^* = 0.3$, determine whether a single seed in one community can propagate to the other. What about $q^* = 0.1$?

5. Show that in a best-response dynamic on a coordination graph game, no cycles can occur (monotone convergence). Use this to bound the convergence time on a graph of size n.

6. Formalize the following as a network coordination game and compute its contagion threshold: a population of 100 poker regulars are each in one of two Discord servers (50 and 50), with complete graphs within each server and 5 bridging edges between. A new sizing convention is introduced in server 1. What is the minimum q^* for which it spreads to server 2?

7. **Argue, structurally,** why contagion on a network identifies the risk-dominant action of a coordination game rather than the payoff-dominant action. Interpret this as a game-theoretic "mechanism design" observation: networks select on risk, not on payoff.

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Concepts: nash-equilibrium, graph-theory, phase-transitions
Prerequisites: 8.1, 2.1
Enables: 8.3, 8.6

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