Best-Shot and Public Goods on Graphs

Game Theory · Module 8 — Network Games

Node 8.4

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Network Games **Node:** 8.4

In the linear-quadratic graph games of 8.1, the interaction parameter ϕ could be positive (complements) or negative (substitutes). This node specializes to the case of **strategic substitutes** — games where a neighbor's effort *reduces* my incentive to exert effort — and the canonical application: **local public goods**. The foundational model is Bramoullé and Kranton (2007): each player on a graph provides a costly effort that benefits themselves and their neighbors; because neighbors' efforts substitute for own effort, equilibrium produces **specialization** — some players provide high effort while their neighbors free-ride. The structure of the interaction graph determines *who* specializes and *who* free-rides, and the equilibrium set is characterized by the **independent sets** and **maximal independent sets** of the graph. This is the game-theoretic foundation of free-riding on networks, division of labor, and the economics of local public goods.

Formal Definition

A **public goods game on a graph** is a graph game $(N, G, \{S_i\}, \{u_i\})$ where each player i chooses an effort level $x_i \in [0, \bar{x}]$ and has payoff

$$u_i(x_i, x_{\mathcal{N}_i}) = f\left(x_i + \sum_{j \in \mathcal{N}_i} x_j\right) - c(x_i),$$

where f is a concave, increasing benefit function and c is a convex, increasing cost function. The benefit depends on the **total effort in i's closed neighborhood** $\bar{\mathcal{N}}_i = \{i\} \cup \mathcal{N}_i$.

Best-shot game. The extreme case of perfect substitutes: $f(y) = y$, but with $c(x_i)$ convex enough that it is optimal to have exactly one provider per neighborhood. In the pure best-shot model:

$$u_i = \max\left(x_i, \max_{j \in \mathcal{N}_i} x_j\right) - c(x_i).$$

Your benefit is the maximum effort in your neighborhood, not the sum. Only the best contributor matters.

Linear-quadratic substitutes. The tractable benchmark from Bramoullé–Kranton:

$$u_i = \alpha\left(x_i + \sum_{j \in \mathcal{N}_i} x_j\right) - \frac{1}{2}\left(x_i + \sum_{j \in \mathcal{N}_i} x_j\right)^2 - \frac{\gamma}{2} x_i^2,$$

where $\gamma > 0$ is a private cost parameter. With the simpler specification:

$$u_i = x_i + \sum_{j \in \mathcal{N}_i} x_j - \frac{1}{2}\left(x_i + \sum_{j \in \mathcal{N}_i} x_j\right)^2,$$

the first-order condition gives the best response

$$x_i^* = \max\left(0,\ 1 - \sum_{j \in \mathcal{N}_i} x_j^*\right).$$

This says: **provide effort up to the point where the total neighborhood effort (including your own) reaches 1, but never go negative**. If your neighbors already provide enough, you provide zero — you free-ride.

Nash equilibrium characterization (Bramoullé–Kranton). In the binary effort version ($x_i \in \{0,1\}$), a Nash equilibrium is a profile where every player with $x_i = 1$ has no neighbor with $x_j = 1$, and every player with $x_i = 0$ has at least one neighbor with $x_j = 1$. The set of providers $P = \{i : x_i = 1\}$ is a **maximal independent set** of G: independent (no two providers are adjacent) and maximal (every non-provider is adjacent to a provider). Conversely, every maximal independent set of G corresponds to a Nash equilibrium.

Intuition

The Bramoullé–Kranton model captures a simple but powerful idea: **effort on a network is a local public good, and local public goods produce specialization**. When my neighbor works hard, I do not need to — I benefit from their work for free. This is strategic substitutability: my neighbor's high effort reduces my marginal benefit of effort, so my best response is to lower my own. Conversely, if my neighbor shirks, I must step up. The equilibrium is a division of labor: some nodes are "providers" and their neighbors are "free-riders."

The graph structure determines which equilibria exist. On a star, the center can be the sole provider (all peripherals free-ride) or a peripheral can be a provider (and the center and others free-ride, but this is less stable). On a cycle, alternating nodes provide — the equilibrium is a coloring. On a random graph, the set of providers is an independent set, and the number of equilibria is the number of maximal independent sets, which can be exponential.

The key structural insight: **the equilibrium correspondence between Nash equilibria and maximal independent sets means that the combinatorial structure of independent sets in graphs governs the economics of local public goods**. Graphs with few maximal independent sets have few equilibria (tight predictions); graphs with many have many equilibria (coordination is hard). Trees have polynomial numbers; dense random graphs have exponential numbers.

Free-riding is not pathological in this model — it is *efficient* in the sense that total effort is minimized for a given level of coverage. The inefficiency comes from a different source: the wrong set of players may specialize. A low-cost provider might free-ride while a high-cost neighbor provides, because the graph topology forced the wrong assignment. This misallocation is the welfare cost of decentralized provision on networks.

Structural Role

8.4 is the strategic-substitutes counterpart to 8.2's coordination (strategic complements). Together they span the two fundamental interaction types in network games.

- **From 8.1:** inherits the graph-game framework; specializes to substitutes ($\phi < 0$).
- **From 8.2/8.6:** contrast with coordination and complementarities — complements produce conformity; substitutes produce specialization.
- **To 8.5:** the efficiency of public-goods equilibria motivates the network-formation question: which graphs produce efficient public-goods outcomes?
- **To Module 6:** mechanism design for public goods on networks — how to design transfers that correct the misallocation from decentralized provision.

Mathematical Development

Best-response characterization

In the linear-quadratic model, write $y_i = \sum_{j \in \mathcal{N}_i} x_j$ for the total neighbor effort. The best response is

$$x_i^*(y_i) = \max(0, 1 - y_i).$$

This is a piece-wise linear, decreasing function of y_i. An interior equilibrium satisfies $x_i = 1 - y_i$ for all active providers (those with $x_i > 0$) and $x_i = 0$ with $y_i \geq 1$ for free-riders.

Equilibrium structure on specific graphs

Complete graph K_n. Any single-provider profile is a Nash equilibrium: one player provides $x = 1$, all others provide 0, everyone gets benefit 1. There are n such equilibria. There is also a symmetric interior equilibrium $x_i = 1/(n)$ for all i, where each player provides a small amount. The single-provider equilibria are maximal independent sets of size 1 (every node is adjacent to every other).

Star $K_{1,n}$. Two types of equilibria: (a) center provides $x_c = 1$, all peripherals provide 0. (b) Any single peripheral p_j provides $x_{p_j} = 1$, center provides 0, all other peripherals provide 0. Type (a) is the "center specializes" equilibrium; type (b) is the "peripheral specializes" equilibrium, but the center still gets the good because p_j is the center's neighbor.

Cycle C_n. For even n: alternating providers (every other node provides 1, rest provide 0). Two such equilibria (offset by one position). For odd n: no perfect alternating equilibrium exists; instead, one "gap" must exist where two adjacent nodes both provide 0, and a third node covers both.

General theorem. The number of Nash equilibria equals the number of maximal independent sets of G, which is bounded above by $3^{n/3}$ (Moon–Moser bound) and can be exponential in n for general graphs.

Welfare analysis

Social optimum. A social planner would choose efforts to maximize total welfare $\sum_i u_i$. In the binary model, the planner wants to find the **minimum dominating set** — the smallest set of providers such that every node is either a provider or adjacent to one. This is NP-hard in general but polynomial on trees and some special graph classes.

Efficiency gap. Nash equilibria (maximal independent sets) are generally **larger** than the minimum dominating set — too many players provide, not too few. The inefficiency of decentralized provision is over-provision by poorly positioned players, not under-provision.

Theorem (Bramoullé–Kranton). On any graph, the Nash equilibria of the public-goods game are Pareto-ranked: the equilibrium with the smallest provider set is Pareto-best, and the one with the largest provider set is Pareto-worst. The efficient equilibrium corresponds to the smallest maximal independent set.

Heterogeneous costs

When players have different costs $c_i(x_i) = \gamma_i x_i^2/2$, the best response becomes $x_i^* = \max(0, (1 - y_i)/(1 + \gamma_i))$. High-cost players are more likely to free-ride; low-cost players are more likely to provide. On a bipartite graph, the efficient equilibrium assigns provision to the low-cost side. On a general graph, the assignment of providers to independent-set positions depends on the interplay between cost structure and graph topology.

Worked Example

Public goods on a path graph

Consider a path graph P_5 with 5 nodes: $1 - 2 - 3 - 4 - 5$. Binary effort, threshold model.

Maximal independent sets: $\{1, 3, 5\}$, $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$. Each is a Nash equilibrium.

Equilibrium $\{1, 3, 5\}$: three providers, two free-riders. Total effort: 3. Every node is covered (within distance 1 of a provider).

Equilibrium $\{2, 4\}$: two providers, three free-riders. Total effort: 2. Nodes 1, 3, 5 are all adjacent to a provider. This is the minimum dominating set and the Pareto-best equilibrium.

Equilibrium $\{1, 4\}$: two providers. Node 1 covers nodes 1 and 2. Node 4 covers 3, 4, 5. Total effort: 2. Also minimum.

Welfare comparison. In the $\{2, 4\}$ equilibrium, total cost is $2c$. In the $\{1, 3, 5\}$ equilibrium, total cost is $3c$. Both achieve full coverage, but the smaller provider set wastes less effort. Decentralized dynamics can get stuck at the larger equilibrium.

Decentralized dynamics. Start with everyone providing ($x_i = 1$ for all i). Under best-response updates: node 2 sees neighbors 1 and 3 providing, so free-rides ($x_2 = 0$). Node 4 sees neighbors 3 and 5 providing, so free-rides. Now $\{1, 3, 5\}$ provide, $\{2, 4\}$ free-ride. This is an equilibrium. The dynamics converged to the Pareto-worst equilibrium because they started from a state with too much provision and shaved off the "unnecessary" providers, but not enough of them.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Effort x_i	Depth of solver work on a particular spot or board texture
Neighbor benefit	Your study partners benefit from your solver output without running it themselves
Free-riding	Relying on study partners' solver work instead of doing your own
Provider (independent set)	The study-group member who actually runs the sims and shares the results
Maximal independent set	The smallest subset of a group such that everyone is adjacent to a provider
Strategic substitutability	"If you already ran this sim, I do not need to"

Concrete situation: solver work in a study group

A 5-person study group works on a shared topic — say, turn play in 3-bet pots out of position. Each person could run their own solver trees, but solver work is expensive (time, compute, mental effort). The output is a local public good: if Alex runs the sim, Alex's study partners (neighbors in the group graph) can read Alex's output and get 80% of the benefit without doing the work.

The equilibrium is specialization: one or two members become the "solver grinders" while the rest free-ride on their output. This is not laziness — it is the Nash equilibrium of a public-goods game with strategic substitutes on the group's interaction graph. The free-riders contribute in other ways (e.g., providing hand histories, running live-play experiments, reviewing results), but the core solver work is provided by a maximal independent set of the group.

The graph structure matters. If the group is fully connected (K_5), a single provider suffices. If the group has subgroups (a 3-person cluster and a 2-person cluster connected by one bridge), the minimum dominating set needs one provider per cluster. The worst outcome is everyone providing — redundant effort. The second-worst is the wrong person providing (a player who is slow at solver work but well-connected).

Concrete situation: coaching ecosystem

In a coaching stable, the head coach produces content (solver analyses, conceptual frameworks, hand reviews). Coaching clients are nodes in a graph where the coach is the hub. The coach's effort is a public good for all clients. If the coach provides, no client needs to independently derive the material — they free-ride. If the coach stops providing, clients must step up or find another provider.

The pairwise stability question from 8.5 connects here: is this coaching graph stable? It is, as long as the coach's effort exceeds the clients' threshold for benefit and the coaching fee is below the clients' willingness to pay for the public good. If another coach opens a competing stable, clients at the boundary face a public-goods decision: which coach's output to free-ride on.

What this understanding buys you

Free-riding in your study group is structurally predicted, not a character flaw. If you find that one person does all the solver work and everyone else reads the results, that is the Nash equilibrium of a substitutes game on your group graph. To change the equilibrium, change the graph: split the group into smaller subgroups where each person must be a provider for at least one topic. Or change the cost structure: make solver work cheaper (shared compute, better tools) so that the marginal cost of providing drops below the marginal benefit of not relying on a single provider.

Your position in the study graph determines your role. If you are the most-connected node (the hub), you will end up as the provider — you have the most neighbors to cover. If you are peripheral, you will free-ride. This is not a choice you make — it is the structural consequence of your graph position. Understanding this lets you predict your role and, if you want to change it, restructure your connections.

Over-provision is the inefficiency, not under-provision. In poker study groups, the typical pathology is not "nobody does the work" — it is "everyone redundantly does the same work." Three people in a group independently run the same solver tree because they do not trust each other's output or do not have efficient sharing mechanisms. This is the $\{1,3,5\}$ equilibrium on the path graph — Pareto-dominated by $\{2,4\}$. The fix is better sharing infrastructure, not more individual effort.

Cross-Links

- **8.1 Graph games** — public goods as the substitutes specialization of linear-quadratic graph games.
- **8.2 Coordination on networks** — contrasting case: complements produce conformity, substitutes produce specialization.
- **8.5 Network formation** — which graphs arise endogenously when agents can form links to access public goods.
- **6.1 Mechanism design** — how to design transfer schemes for efficient public-goods provision on networks.
- **2.1 Nash equilibrium** — maximal-independent-set characterization of equilibria.
- **Graph theory** — independent sets, dominating sets, chromatic number as game-theoretic solution concepts.

Structural Compression

Invariant: In a local public-goods game with strategic substitutes, Nash equilibria correspond to maximal independent sets of the interaction graph; the smallest such set is Pareto-best, and free-riding is the structural consequence of substitutability, not a behavioral pathology.

Mechanism: Effort is a local public good benefiting the closed neighborhood; best response is decreasing in neighbors' effort (substitutes); equilibrium produces specialization where providers form an independent set and non-providers free-ride; efficiency requires the minimum dominating set, which is generically smaller than the equilibrium provider set.

Cross-System Echo: The Bramoullé–Kranton structure is the game-theoretic analogue of **distributed resource allocation** in computer networks (where servers provide a shared resource and clients consume it), **coloring problems** in combinatorial optimization (independent sets are dual to proper colorings), and **division of labor** in organizational economics (team members specialize in complementary tasks). The common structure: a shared benefit, a private cost, and a graph that determines who benefits from whose provision. The equilibrium is always specialization, and the efficiency of specialization is always governed by the graph's independent-set structure.

Exercises

1. Enumerate all maximal independent sets of the cycle C_6 and verify that each corresponds to a Nash equilibrium of the binary public-goods game. Identify the Pareto-best equilibrium.
2. Show that on a tree T, the minimum dominating set can be computed in polynomial time via dynamic programming. Derive the recurrence.
3. For the linear-quadratic public-goods game on K_n, compute the symmetric interior equilibrium and compare total welfare to the single-provider equilibrium. Show that the single-provider equilibrium is Pareto-superior for sufficiently large n.
4. Construct a graph on 8 nodes where the number of maximal independent sets is maximized. How close can you get to the Moon–Moser bound $3^{8/3} \approx 6.16$?
5. In a study group of 6 players on a cycle graph, each player independently decides whether to run a solver sim (cost $c = 0.4$, benefit 1 to self and neighbors). Find all Nash equilibria and compute total welfare for each.
6. Add heterogeneous costs to the public-goods game on a star $K_{1,4}$: the center has cost $\gamma_c = 0.5$, peripherals have cost $\gamma_p = 1.0$. Derive the equilibrium and show that the low-cost center is more likely to be the provider.
7. **Argue, structurally,** why the inefficiency of Nash equilibria in local public-goods games comes from over-provision (too many providers) rather than under-provision, and explain how graph topology determines the severity of this inefficiency.

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Concepts: graph-theory, nash-equilibrium, collective-action
Prerequisites: 8.1, 2.1
Enables: 8.5, 8.6

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