Strategic Complementarities

Game Theory · Module 8 — Network Games

Node 8.6

Position in Knowledge Graph

System: Game Theory **Structural Domain:** Network Games **Node:** 8.6

Strategic complementarities are the structural force behind coordination, contagion, and tipping points in network games. An action exhibits strategic complementarity when each player's incentive to take that action *increases* with the number (or proportion) of neighbors who take it. This is the formal content of "doing what others do is rational" – the property that generates bandwagon effects, technology adoption cascades, bank runs, and coordination failures. Nodes 8.1-8.5 introduced games on graphs, coordination, diffusion, public goods, and network formation. Node 8.6 synthesizes these with a unified mathematical treatment grounded in the theory of **supermodular games** (Topkis 1979, Milgrom-Roberts 1990, Vives 1990). The key results are: supermodular games always have pure-strategy Nash equilibria (by Tarski's fixed-point theorem), the equilibrium set has a lattice structure with greatest and least elements, and comparative statics are monotone – an increase in any player's incentive to act raises everyone's equilibrium action. These properties make supermodular games the analytically tractable workhorse of applied game theory, especially in network and industrial-organization contexts.

Formal Definition

Supermodular game. A game $G = (N, \{S_i\}, \{u_i\})$ is **supermodular** if each S_i is a complete lattice and the payoff functions satisfy:

1. **Supermodularity in own action.** For each i and each fixed s_{-i}, $u_i(\cdot, s_{-i})$ is supermodular on S_i:

$$u_i(s_i \vee s_i', s_{-i}) + u_i(s_i \wedge s_i', s_{-i}) \geq u_i(s_i, s_{-i}) + u_i(s_i', s_{-i})$$

   for all $s_i, s_i' \in S_i$, where $\vee$ and $\wedge$ are the lattice join and meet.

2. **Increasing differences (strategic complementarity).** For each i, u_i has increasing differences in (s_i, s_{-i}): if $s'_{-i} \geq s_{-i}$ (in the product-lattice order), then

$$u_i(s_i', s'_{-i}) - u_i(s_i, s'_{-i}) \geq u_i(s_i', s_{-i}) - u_i(s_i, s_{-i})$$

   for all $s_i' \geq s_i$.

Equivalently, for smooth payoffs on $\mathbb{R}$, strategic complementarity is $\partial^2 u_i / \partial s_i \partial s_j \geq 0$ for all $j \neq i$: increasing others' actions raises the marginal return to one's own action.

Strategic complementarity on a network. In a network game on graph $G = (N, E)$, player i's payoff depends on their own action s_i and the actions of their neighbors $\mathcal{N}_i$. Strategic complementarity on the network means

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j} \geq 0 \quad \text{for all } j \in \mathcal{N}_i.$$

The canonical example is the **linear-quadratic network game**:

$$u_i(s_i, s_{-i}) = a_i s_i - \frac{1}{2}s_i^2 + \lambda \sum_{j \in \mathcal{N}_i} g_{ij} s_i s_j,$$

where a_i is a standalone benefit, $\lambda > 0$ is the complementarity parameter, and g_{ij} is the edge weight. The best response is linear:

$$s_i^* = a_i + \lambda \sum_{j \in \mathcal{N}_i} g_{ij} s_j,$$

which in matrix form is $s^* = a + \lambda G s^*$, giving the Nash equilibrium $s^* = (I - \lambda G)^{-1} a$ when λ is small enough for convergence.

Strategic substitutes. The opposite: $\partial^2 u_i / \partial s_i \partial s_j \leq 0$. Each player's incentive to act *decreases* with neighbors' actions. Free-riding and anti-coordination games exhibit strategic substitutes.

Intuition

Strategic complementarity formalizes the idea that "doing more is more attractive when others do more." This positive feedback loop is the engine of coordination games, technology adoption, social conventions, and financial contagion. The formal consequence is that the game has clean analytical properties: equilibria exist, they are ordered, and they respond monotonically to parameter changes. This makes supermodular games the first-choice model for any applied setting where coordination effects are present.

The contrast with strategic substitutes is sharp. Under substitutes, the feedback is negative: doing more is less attractive when others do more. This produces anti-coordination (Hawk-Dove, free-riding on public goods) rather than coordination. The equilibrium structure under substitutes is different: uniqueness is more common, and the comparative statics are ambiguous.

On networks, strategic complementarity interacts with the graph structure to produce the phenomena studied in 8.2 (coordination), 8.3 (cascades), and 8.4 (public goods). High-degree nodes (hubs) have more neighbors, so their complementarity effect is amplified: a hub's action influences more neighbors, and more neighbors influence the hub. This produces the **key-player effect** (Ballester-Calvo-Armengol-Zenou 2006): in a network game with complementarities, the player whose removal most reduces total equilibrium activity is the one with the highest "intercentrality" – a network centrality measure derived from the equilibrium formula $s^* = (I - \lambda G)^{-1} a$.

The Ballester et al. result connects network game theory to policy: if you want to reduce total crime in a network (where crime exhibits complementarities), removing the key player (the one with the highest intercentrality, not necessarily the highest degree or the most active criminal) has the largest effect. This is a non-obvious conclusion that requires the full supermodular-game-on-network machinery.

Structural Role

8.6 is the capstone of Module 8, providing the unified mathematical framework that connects 8.1-8.5:

- **8.1 (Graph games)** introduced the network structure; 8.6 adds the strategic complementarity that makes the structure matter analytically.
- **8.2 (Coordination on networks)** is the special case of binary-action supermodular games on graphs.
- **8.3 (Diffusion and cascades)** is the dynamic version: cascades happen because complementarity creates tipping points.
- **8.4 (Public goods on graphs)** can exhibit either complementarity or substitutability depending on the production function.
- **8.5 (Network formation)** asks which networks are formed strategically; 8.6 analyzes play on a given network.

Beyond Module 8, strategic complementarities connect to:

- **9.5 (Convergence to equilibrium)** – supermodular games are the class where learning dynamics are best-behaved (monotone convergence to the equilibrium set).
- **10.1 (Games embedded in institutions)** – institutions often create complementarities by design.
- **Statistics (Ising models)** – the Ising model on a graph is a supermodular game with binary strategies and nearest-neighbor complementarities.

Mathematical Development

Tarski's fixed-point theorem and equilibrium existence

Theorem (Tarski 1955). If L is a complete lattice and $f : L \to L$ is an order-preserving (monotone) function, then the set of fixed points of f is nonempty and itself forms a complete lattice.

Application to supermodular games. Define the joint best-response correspondence $B(s) = (B_1(s_{-1}), \ldots, B_n(s_{-n}))$. Under strategic complementarity, B is monotone: if $s' \geq s$, then $B(s') \geq B(s)$. By Tarski's theorem, the set of Nash equilibria (fixed points of B) is nonempty and forms a complete lattice with a greatest element $\bar{s}$ and a least element $\underline{s}$.

The greatest and least equilibria are reached by iterated best response from the top and bottom of the strategy space respectively. This gives a constructive algorithm for finding them and establishes that best-response dynamics from any initial condition converge to the equilibrium set.

Monotone comparative statics

Theorem (Milgrom-Roberts 1990). In a supermodular game, if a parameter t increases the marginal return to each player's action (formally: u_i has increasing differences in (s_i, t)), then the greatest and least equilibria are weakly increasing in t.

This is the formal version of "if incentives increase, equilibrium actions increase." It is the reason supermodular games are the workhorse of applied theory: you can do comparative statics without solving for the full equilibrium – just check the direction of the parameter change and the complementarity structure.

The Ballester-Calvo-Armengol-Zenou model

In the linear-quadratic network game, the Nash equilibrium is $s^* = (I - \lambda G)^{-1} a$, which can be expanded as a power series:

$$s_i^* = a_i + \lambda \sum_j g_{ij} a_j + \lambda^2 \sum_j \sum_k g_{ij} g_{jk} a_k + \cdots$$

This is a **Katz-Bonacich centrality** measure: player i's equilibrium action is proportional to a weighted count of all walks of all lengths emanating from i in the network, with longer walks discounted by powers of λ. The equilibrium action is higher for more central players (by the Bonacich centrality measure).

Key-player identification. The key player is the one whose removal causes the largest drop in total equilibrium action $\sum_i s_i^*$. Ballester et al. show this is the player with the highest **intercentrality**, defined as

$$c_i^* = \frac{b_i(G, \lambda)^2}{m_{ii}(G, \lambda)},$$

where b_i is the Bonacich centrality and m_{ii} is the (i, i)-entry of $(I - \lambda G)^{-1}$.

Convergence of learning dynamics in supermodular games

Theorem (Milgrom-Roberts 1990). In a finite supermodular game, iterated elimination of dominated strategies yields the same result from the top and bottom of the strategy space: the greatest and least surviving strategy profiles are the greatest and least Nash equilibria. Moreover, any learning dynamics that are "adaptive" (each player's strategy is eventually a best response to some belief in the range of opponents' recent play) converge to the interval $[\underline{s}, \bar{s}]$.

This is one of the strongest convergence results in game theory. It says that in supermodular games, strategic complementarity pins down the learning dynamics: they always converge to the equilibrium set. This is the connection to Module 9: supermodular games are the "well-behaved" class where learning works.

Binary-action supermodular games and thresholds

When actions are binary ($s_i \in \{0,1\}$), strategic complementarity reduces to: each player i has a **threshold** k_i such that they choose action 1 iff at least k_i of their neighbors choose 1. This is the threshold model of 8.2 and 8.3, now derived as a special case of the supermodular framework. The equilibria are the fixed points of the threshold dynamics; the greatest and least equilibria are the "maximal coordination" and "minimal coordination" outcomes.

Worked Example

Linear-quadratic game on a star network

Network. 5 players: a hub (player 1) connected to 4 spokes (players 2-5). No spoke-to-spoke connections.

Payoffs. $u_i = as_i - s_i^2/2 + \lambda s_i \sum_{j \in \mathcal{N}_i} s_j$, with $a = 1$ for all players, $\lambda = 0.2$.

Best responses. Hub: $s_1^* = 1 + 0.2(s_2 + s_3 + s_4 + s_5)$. Spoke i: $s_i^* = 1 + 0.2s_1$.

Equilibrium. By symmetry, spokes are identical: $s_2 = s_3 = s_4 = s_5 \equiv s_{\text{sp}}$. Hub: $s_1 = 1 + 0.8s_{\text{sp}}$. Spoke: $s_{\text{sp}} = 1 + 0.2s_1 = 1 + 0.2(1 + 0.8s_{\text{sp}}) = 1.2 + 0.16s_{\text{sp}}$. So $0.84s_{\text{sp}} = 1.2$, $s_{\text{sp}} \approx 1.429$. Hub: $s_1 = 1 + 0.8 \times 1.429 \approx 2.143$.

Bonacich centrality. The hub's equilibrium action (2.143) is much larger than each spoke's (1.429), reflecting the hub's higher centrality. If we remove the hub, the spokes become isolated and each plays $s_i^* = 1$. Total action drops from $2.143 + 4 \times 1.429 = 7.86$ to $4 \times 1 = 4$. Removing a spoke instead: remaining network has hub + 3 spokes. New spoke equilibrium: $s'_{\text{sp}} = 1 + 0.2s'_1$, $s'_1 = 1 + 0.6s'_{\text{sp}}$. Solving: $s'_{\text{sp}} \approx 1.363$, $s'_1 \approx 1.818$. Total: $1.818 + 3 \times 1.363 = 5.91$. Drop: $7.86 - 5.91 = 1.95$.

Removing the hub drops total action by 3.86; removing a spoke drops it by 1.95. The hub is the key player – its intercentrality is highest. Policy implication: targeting the hub has roughly double the impact of targeting a spoke.

Comparative statics: increasing the complementarity parameter

Increase λ from 0.2 to 0.3. New spoke equilibrium: $s_{\text{sp}} = 1 + 0.3s_1$, $s_1 = 1 + 1.2s_{\text{sp}}$. $s_{\text{sp}} = 1 + 0.3(1 + 1.2s_{\text{sp}}) = 1.3 + 0.36s_{\text{sp}}$. $0.64s_{\text{sp}} = 1.3$, $s_{\text{sp}} \approx 2.031$. $s_1 \approx 3.438$. Total: $3.438 + 4 \times 2.031 = 11.56$.

As λ increases, everyone's equilibrium action increases – monotone comparative statics, exactly as the theory predicts. The amplification is stronger at the hub because the hub has more neighbors to amplify through.

Poker Anchor

Concept mapping

Formal object	Poker instantiation
Strategic complementarity	The more your opponents bluff, the more profitable it is for you to bluff-catch (and vice versa)
Strategic substitutability	The more your opponents value-bet, the less profitable it is for you to value-bet the same spot
Supermodular structure	Games where aggressive play breeds aggressive response – escalation dynamics
Network complementarity	In a table with regulars, one player's aggression raises all others' aggression through strategic response
Key player (hub)	The player whose removal most changes the table dynamic – the meta-setter
Bonacich centrality	A player's "influence" measured by how many strategic interactions they mediate

Concrete situation: table dynamics driven by one aggressive player

A 6-max table has five balanced regulars and one hyper-aggressive LAG. The LAG's aggression creates complementarity: the regulars' best response to the LAG is to widen their calling ranges, which in turn makes the LAG's bluffs less profitable, which adjusts the LAG's strategy, and so on. The equilibrium of the full table is determined by the complementarity structure: the LAG is the hub, the regulars are the spokes, and the equilibrium aggression levels are higher than if the LAG were replaced by a balanced player.

If the LAG leaves the table, the entire table dynamic shifts: everyone tightens up, average pot size drops, and the aggression level decreases. The LAG is the key player – their removal has the largest effect on total table action. This is the Ballester et al. result applied to the poker table.

What this understanding buys you

Complementarity explains why poker metas are "sticky." Once a table converges to a high-aggression equilibrium (everyone 3-betting light), it stays there because any individual deviation to tightness is punished by the complementarity – your opponents are already playing aggressively, so tightening up costs you pots. The complementarity creates a coordination trap: everyone would prefer lower aggression (more readable play), but no one can unilaterally move there. This is the supermodular game's greatest-equilibrium trap.

Key-player removal is the fastest way to shift a meta. If you want to change a table's dynamic (e.g., from hyper-aggressive to balanced), the most efficient intervention is removing the key player – the one whose style has the most amplified effect through the complementarity structure. In practice, this means: if the LAG leaves, the table calms down much faster than if a nit leaves.

Cross-Links

- **8.1 Graph games** – the network structure on which complementarities operate.
- **8.2 Coordination on networks** – binary-action complementarities as a special case.
- **8.3 Diffusion and cascades** – complementarity-driven tipping points.
- **8.4 Public goods on graphs** – complementarity vs substitutability in provision.
- **9.5 Convergence to equilibrium** – supermodular games are where learning converges cleanly.
- **10.1 Games embedded in institutions** – institutional complementarities.
- **Statistics (Ising models)** – binary supermodular games as lattice spin models.
- **Economics (industrial organization)** – Cournot and Bertrand games with complementary products.

Structural Compression

Invariant: In supermodular games (games with strategic complementarities), the Nash equilibrium set is nonempty and forms a complete lattice with greatest and least elements; comparative statics are monotone (increasing parameters raise equilibrium actions); and adaptive learning dynamics always converge to the equilibrium set. On networks, complementarities amplify through the graph structure, with the key player determined by Bonacich intercentrality.

Mechanism: Increasing differences in the payoff function ensure that the best-response correspondence is monotone; Tarski's fixed-point theorem gives existence and lattice structure; the power-series expansion $(I - \lambda G)^{-1}a$ in linear-quadratic network games links equilibrium actions to walk-based centrality measures; comparative statics follow from monotone selection within the lattice.

Cross-System Echo: Strategic complementarity is the game-theoretic formalization of **positive feedback** in systems theory and **ferromagnetic interaction** in statistical physics. The Ising model on a graph with ferromagnetic couplings is a binary supermodular game; the Gibbs measure is the stochastic-stability-selected equilibrium; the phase transition (spontaneous magnetization) is the cascade threshold of 8.3. The lattice structure of equilibria mirrors the lattice of Gibbs states in statistical mechanics. The connection is not metaphorical – the mathematics is identical, and insights from one field transfer directly to the other.

Exercises

1. Verify that the linear-quadratic network game with $\lambda > 0$ satisfies the strategic complementarity condition (positive cross-partial derivatives). Then verify supermodularity.
2. For the star network with 5 nodes from the worked example, compute the Nash equilibrium for $\lambda = 0.1, 0.2, 0.3$ and verify the monotone comparative statics result.
3. Compute the Bonacich centrality vector for a 4-node cycle graph with uniform edge weights and $\lambda = 0.15$. Identify the key player (by symmetry, all are equivalent – verify this).
4. In a binary-action supermodular game on a complete graph (K_n), derive the threshold model: each player adopts action 1 iff at least k others do, where k depends on the payoff parameters. Characterize all Nash equilibria.
5. Show that in a game with strategic substitutes ($\partial^2 u_i / \partial s_i \partial s_j \leq 0$), the best-response correspondence is decreasing, and Tarski's theorem still applies (the equilibrium set is nonempty) but the greatest and least equilibria have different comparative statics. Verify with a Cournot duopoly example.
6. For the linear-quadratic game on a random Erdos-Renyi graph $G(n, p)$ with $n = 20, p = 0.2, \lambda = 0.1$, compute (numerically) the equilibrium and identify the key player. Verify that the key player has the highest intercentrality, not necessarily the highest degree.
7. **Argue, structurally,** why supermodular games are the "well-behaved" class of games for learning and dynamics. What specific property of the best-response correspondence makes convergence guaranteed, and why does this fail for general games?

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Concepts: nash-equilibrium, fixed-point, graph-theory
Prerequisites: 8.1, 8.2, 8.5
Enables: 9.5, 10.1

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