Lesson 1.1 — Addition & Subtraction: Combining and Undoing Relations

Mathematical Intuition · Module 1 — Operations

Node 1.1

1. Why this matters (Hook)

Addition and subtraction look like the most "basic" operations, so we tend to stop thinking about them early in life.

But almost every advanced equation — in physics, probability, quantum mechanics, optimization — is built from **clever ways of adding and subtracting**:

- adding energies, probabilities, or costs
- subtracting baselines, references, or "what we already know"
- forming differences that measure change

If you see **what is being combined** and **what is being undone**, almost any expression becomes readable.

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2. Core Intuition

2.1 What Addition Really Is

At its core, **addition is "putting relationships side by side."**

We are not just stacking counts. We are combining *effects* or *contributions*:

- "this much **and** that much"
- "initial state **plus** change"
- "force from source A **plus** force from source B"

Key images:

- Two arrows (vectors) placed head-to-tail → one longer arrow.
- Two flows of water merging into one stream.
- Two "stories" about reality contributing to the same total effect.

So when you see:

$$z = x + y$$

read:

> "z is the result of the **joint contribution** of x and y."

The important question is **not** "what are the numbers?" but:

"What *kind* of contribution is x?
What *kind* of contribution is y?"

2.2 What Subtraction Really Is

Subtraction is **un-combining**:

- remove a contribution
- compare two states
- isolate what changed

Three main roles:

1. **Difference**

$$a - b$$

"How much more is a than b?"
Difference = *distance between states.*

2. **Change**

$$\Delta x = x_{\text{final}} - x_{\text{initial}}$$

"What happened to x?"
Change = *final minus baseline.*

3. **Undoing**
If

$$z = x + y,$$

then

$$z - y = x$$

means "remove the y-contribution to recover x."

Subtraction is how we **separate** influences that were previously combined.

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3. Formal Lens (Light)

Let's look at a few standard forms and read them as sentences.

3.1 Initial + Change

$$x(t) = x_0 + vt$$

- x_0: initial position
- vt: change due to constant velocity over time

Sentence:

"Position at time t = **where you started + how far the constant motion has carried you**."

3.2 Sum of Contributions

$$F_{\text{net}} = F_1 + F_2 + \cdots + F_n$$

Sentence:

"Net force is the **combined effect** of all individual forces acting on the object."

Each F_i is a separate relationship with the object; addition lays them side by side.

3.3 Difference as Measurement

$$V = V_{\text{high}} - V_{\text{low}}$$

Sentence:

> "Voltage is **how much higher** one electric potential is compared to another."

We don't care about absolute values; **the subtraction defines what is physically meaningful**.

3.4 Error / Residual

$$\text{error}_i = y_i - \hat{y}_i$$

Sentence:

> "Error is the difference between what happened (y_i) and what we predicted ($\hat{y}_i$)."

Subtraction is what turns two separate stories into a *mismatch*.

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4. Cross-Domain Examples

4.1 Physics — Energy and Work

Work done by a constant force:

$$W = F \cdot d$$

But energy bookkeeping often looks like:

$$E_{\text{total}} = K + U$$

- K: kinetic energy (motion)
- U: potential energy (position / configuration)

Sentence:

> "Total energy is **motion contribution** plus **configuration contribution**."

When energy is "lost," we often mean:

$$\Delta E = E_{\text{final}} - E_{\text{initial}} < 0$$

Subtraction measures **net change** in the system.

4.2 Probability — Law of Total Probability

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$$

Sentence:

> "Chance of A is the sum of its contributions from two disjoint scenarios: one where B happens, one where B does not."

Addition here is summing **disjoint stories** about how A can happen.

4.3 Quantum Mechanics — Superposition

A 2-state superposition:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Addition here is **not** just "0 plus 1."
It is combining *possibility patterns*:

> "The state is the superposition of two basis possibilities, weighted by amplitudes α and β."

Subtraction appears when you form interference terms or differences of amplitudes, e.g. $\alpha - \beta$, to see how two paths oppose each other.

4.4 Optimization — Loss Decomposition

A regularized loss:

$$L(\theta) = \underbrace{\sum_i \ell(f_\theta(x_i), y_i)}_{\text{data mismatch}} + \underbrace{\lambda R(\theta)}_{\text{regularization penalty}}$$

Sentence:

> "Total loss is the **sum of mistakes on data** plus a **penalty** for complicated parameter configurations."

Addition combines two qualitatively different influences: fitting data and keeping the model tame.

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5. Micro Drills

Use these as quick checks; answers can be in words.

1. **Role identification**
   For each equation, say whether the main role of + or − is:
   - combining contributions
   - measuring difference
   - expressing change / update

   a) $s = s_0 + vt$

   b) $\Delta T = T_{\text{final}} - T_{\text{initial}}$

   c) $I_{\text{total}} = I_1 + I_2 + I_3$

2. **Rewrite in "initial + change" form**

   Rewrite each sentence as an equation in the pattern:

   $$\text{final} = \text{initial} + \text{change}$$

   a) "The bank balance after interest is the original balance plus the interest earned."

b) "The new temperature is the old temperature minus what was lost to cooling."

3. **Difference vs absolute**

   You measure two voltages: $V_1 = 5\text{V}$, $V_2 = 8\text{V}$.
   Which is physically more meaningful in a circuit: (V_1, V_2) separately, or $V_2 - V_1$?
   Explain in one sentence.

4. **Error interpretation**

   Given $\text{error}_i = y_i - \hat{y}_i$,
   what does "error = 0" mean?
   what does a large positive error mean?
   what does a large negative error mean?

5. **Quantum superposition**

   In $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,
   what is the addition symbol conceptually doing?
   (Answer in one sentence, no formulas.)

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6. Reflection

- When you see a + or − in an equation from now on, ask:
  - "What are the *stories* being combined?"
  - "Is this a sum of influences, or a difference of states?"
- Look at one equation from your own work (VQE, regularization, or physics).
  Without computing anything, write one short paragraph that explains:
  - what is being added,
  - what is being subtracted,
  - and why it makes sense conceptually.
