Lesson 3.2 — Rationals: Dense But Full of Holes

Mathematical Intuition · Module 3 — Numbers Spaces

Node 3.2

1. Why this matters (Hook)

The rational numbers $\mathbb{Q}$ are the first number system where division always works (except by zero).

They seem to fill the number line: between any two rationals, you can always find another. In fact, you can find infinitely many others. The rationals are **dense** — there are no gaps between them.

And yet, the rationals are riddled with holes.

$\sqrt{2}$ is not rational. π is not rational. e is not rational.

The rationals are an infinite, dense set that still manages to miss **almost all** real numbers.

Understanding this paradox — dense but incomplete — is the key to understanding why calculus needs the reals, why computers struggle with exact arithmetic, and why approximation is always a trade-off.

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2. Core Intuition

2.1 Rationals are exact ratios

$$\frac{p}{q}, \quad p \in \mathbb{Z}, \quad q \in \mathbb{Z} \setminus \{0\}$$

A rational number is the answer to a division that works out exactly. "Three-fourths" means: divide something into 4 equal parts, take 3.

Every integer is rational ($n = n/1$). The rationals extend the integers by allowing exact division.

2.2 Density: between any two, always another

Between $\frac{1}{2}$ and $\frac{3}{4}$:

$$\frac{1}{2} < \frac{5}{8} < \frac{3}{4}$$

Between $\frac{1}{2}$ and $\frac{5}{8}$:

$$\frac{1}{2} < \frac{9}{16} < \frac{5}{8}$$

This never stops. You can always average two rationals to get a rational between them: $\frac{a+b}{2}$.

On the number line, rationals are everywhere. There is no interval, no matter how small, that is free of rationals.

2.3 The holes: what density misses

Despite being everywhere, the rationals skip certain points.

Claim: There is no rational number whose square is 2.

Proof sketch: Suppose $(p/q)^2 = 2$ with p/q in lowest terms. Then $p^2 = 2q^2$, so p^2 is even, so p is even. Write $p = 2k$, then $4k^2 = 2q^2$, so $q^2 = 2k^2$, so q is even. But then p/q was not in lowest terms — contradiction.

So $\sqrt{2}$ sits on the number line but between all the rational points, in a gap that no fraction can fill. The rationals are dense but **not complete**.

2.4 Decimal expansions as a diagnostic

Every rational number has a decimal expansion that either terminates or repeats:

- $\frac{1}{4} = 0.25$ (terminates)
- $\frac{1}{3} = 0.333\ldots$ (repeats)
- $\frac{22}{7} = 3.142857142857\ldots$ (repeats with period 6)

Conversely, any terminating or repeating decimal is rational.

Non-repeating, non-terminating decimals are irrational: $\pi = 3.14159265\ldots$ never settles into a pattern.

This gives you a test: **if the decimal pattern eventually repeats, the number is rational.**

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3. Formal Lens (Light)

3.1 The rationals as a field

$\mathbb{Q}$ is a **field**: a set with addition, subtraction, multiplication, and division (except by zero), all obeying the standard algebraic laws.

$$(\mathbb{Q}, +, \times) \text{ with } a^{-1} = 1/a \text{ for } a \neq 0$$

Sentence: "In the rationals, every nonzero element has a multiplicative inverse. You can always undo multiplication."

The integers are a ring (no general division). The rationals upgrade to a field (division always works). This is why rational arithmetic is **exact**: there is no rounding.

3.2 Continued fractions: the best rational approximations

Any real number α can be written as a continued fraction:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Truncating at each level gives the **best** rational approximation for the denominator size:

Truncation	Approximation of π	Error
$[3]$	3	0.14159
$[3; 7]$	22/7	0.00126
$[3; 7, 15]$	333/106	0.00003

Truncation	Approximation of π	Error
$[3; 7, 15, 1]$	$355/113$	0.0000003

Sentence: "Continued fractions find the rational numbers that punch above their weight — small denominators, tiny errors."

$355/113$ approximates π to 6 decimal places with a denominator of only 113. No fraction with a smaller denominator does better.

3.3 Countability: rationals are listable

Despite being dense, the rationals are **countable** — they can be put in a one-to-one correspondence with the natural numbers.

Cantor's diagonal arrangement:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \ldots$$

Sentence: "You can list every rational number in a sequence without missing any. The rationals are infinite but no bigger than the integers."

This is in stark contrast to the reals, which are **uncountable**. The rationals are dense, but they are actually a "thin" set — in measure-theoretic terms, the rationals have measure zero.

3.4 Diophantine approximation: how close can rationals get?

For any irrational α, there are infinitely many rationals p/q satisfying:

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Sentence: "Rationals can approximate any irrational number with error that shrinks quadratically in the denominator. The better the approximation you want, the larger the denominator you need."

This is Dirichlet's theorem. It means rationals are excellent approximators — good enough for engineering, not quite enough for analysis.

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4. Cross-Domain Examples

4.1 Computer Science — floating-point arithmetic

$$\mathrm{fl}(x) = m \times 2^e, \quad m \in \mathbb{Q}$$

Reading: "Floating-point numbers are a finite subset of the rationals (specifically, those with denominators that are powers of 2). Every stored number is rational, but not every rational is representable."

This is why $0.1 + 0.2 \neq 0.3$ in most programming languages: 0.1 is not exactly representable in binary floating-point. The gap between rationals and reals becomes a gap between stored values and mathematical truth.

4.2 Music — just intonation vs equal temperament

Just intonation uses exact rational frequency ratios (3:2 for a fifth, 5:4 for a major third). Equal temperament uses $2^{k/12}$, which is irrational for most k.

Reading: "Rational tuning sounds pure in the key it is built for but goes out of tune in other keys. Irrational (equal temperament) tuning is slightly impure everywhere but equally impure, enabling modulation between keys."

The trade-off between rational exactness and irrational flexibility is literally audible.

4.3 Probability — discrete distributions are rational

$$P(X = k) = \frac{\binom{n}{k}}{2^n}$$

Reading: "In a fair coin-flip experiment, every probability is a rational number. Discrete probability spaces with integer-valued outcomes always produce rational probabilities."

The irrationals only enter probability when you take limits (continuous distributions, the central limit theorem).

4.4 Quantum Computing — phase estimation

$$\hat{U}|\psi\rangle = e^{2\pi i\theta}|\psi\rangle, \quad \theta \in [0, 1)$$

Reading: "Quantum phase estimation approximates the eigenphase θ as a rational number $j/2^n$ using n qubits. More qubits give a finer rational approximation."

The algorithm literally converts an irrational quantum phase into its best rational approximation.

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5. Micro Drills

Answer in words where possible.

1. **Density check:** Find a rational number between $\frac{7}{10}$ and $\frac{8}{11}$. Then find another between those two. Could you keep going forever?
2. **Irrationality proof:** The proof that $\sqrt{2}$ is irrational uses contradiction. What assumption is contradicted? What property of integers does the proof exploit?
3. **Decimal diagnostic:** Is $0.123123123\ldots$ rational? Is $0.101001000100001\ldots$ rational? How do you know?
4. **Approximation quality:** $22/7 \approx 3.14286$ and $355/113 \approx 3.14159292$. Why is $355/113$ considered a much better approximation, even though both are "close" to π?
5. **Closure reflection:** The integers lack closure under division. The rationals fix this. What property do the rationals lack that the reals fix?

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6. Reflection

The rationals are a paradox: they are everywhere on the number line (dense) yet they miss almost everything (measure zero).

In one paragraph, explain why this matters for computation. Specifically: why does the fact that computers store rationals (floating-point numbers) mean that every computed answer is an approximation? What does the density of rationals guarantee about the quality of these approximations?

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Prerequisites: 1.3, 3.1
Enables: 3.3, 3.4

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