Lesson 3.4: Vectors & Inner Products

Mathematical Intuition · Module 3 — Numbers Spaces

Node 3.4

Overview

Understanding vectors as directions with magnitude and inner products as measures of alignment and projection.

Learning Objectives

- See vectors as arrows in space with direction and magnitude
- Understand inner products as measuring alignment
- Build geometric intuition for orthogonality and projection
- Recognize vector spaces and their properties

Core Concepts

What Are Vectors?

Geometric View

- **Arrow in space**: direction + magnitude
- **Position vector**: from origin to point
- **Displacement vector**: from one point to another

Algebraic View

- **Column of numbers**: $\mathrm{v} = [\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n]^T$
- **Linear combination**: $\mathrm{v} = \mathrm{v}_1\mathrm{e}_1 + \mathrm{v}_2\mathrm{e}_2 + \ldots + \mathrm{v}_n\mathrm{e}_n$
- **Basis vectors**: $\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n$ (standard basis)

Vector Operations

Addition

- $\mathbf{u} + \mathbf{v}$: tip-to-tail (parallelogram rule)
- Component-wise: $[\mathrm{u}_1 + \mathrm{v}_1, \mathrm{u}_2 + \mathrm{v}_2, \ldots]^T$
- Commutative: $\mathrm{u} + \mathrm{v} = \mathrm{v} + \mathrm{u}$

Scalar Multiplication

- $\alpha \cdot \mathbf{v}$: scales magnitude by $|\alpha|$, flips direction if $\alpha < 0$
- Component-wise: $[\alpha \cdot \mathrm{v}_1, \alpha \cdot \mathrm{v}_2, \ldots]^T$

Linear Combinations

- $\mathbf{w} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v}$: span of u and v
- Span: all possible linear combinations

Inner Products

Dot Product (Euclidean)

- $\mathbf{u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n}$: sum of component products
- $\mathbf{u \cdot v = ||u|| \cdot ||v|| \cdot \cos(\theta)}$: geometric interpretation
- Measures alignment: positive if acute angle, negative if obtuse

Properties

- **Symmetric**: $u \cdot v = v \cdot u$
- **Linear**: $(\alpha u + \beta v) \cdot w = \alpha(u \cdot w) + \beta(v \cdot w)$
- **Positive definite**: $v \cdot v \geq 0$, with equality iff $v = 0$

Orthogonality

- $\mathbf{u \perp v}$: "u and v are orthogonal" (perpendicular)
- $\mathbf{u \cdot v = 0}$: inner product is zero
- Orthogonal vectors are independent

Norms and Distance

Euclidean Norm

- $\mathbf{||v||_2 = \sqrt{}(v \cdot v) = \sqrt{}(v_1{}^2 + v_2{}^2 + \ldots + v_n{}^2)}$: length of vector
- $\mathbf{||v|| = 0}$ iff $v = 0$

Unit Vectors

- $\mathbf{v = v/||v||}$: unit vector in direction of v
- $\mathbf{||v|| = 1}$: normalized

Distance

- **d(u,v) = ||u - v||**: distance between u and v
- Metric: satisfies triangle inequality

Projections

Projection onto Vector

- **proj_u(v) = (v · u/u · u) · u**: component of v in direction of u
- **v = proj_u(v) + perp_u(v)**: orthogonal decomposition

Projection onto Subspace

- **proj_W(v)**: closest point in subspace W to v
- Minimizes $||v - w||$ over all $w \in W$

Examples in Context

Physics

- **Force vectors**: $F = F_x\hat{\imath} + F_\gamma\hat{\jmath} + F_u k$
- **Work**: $W = F \cdot d$ (force dot displacement)
- **Torque**: $\tau = r \times F$ (cross product)

Machine Learning

- **Feature vectors**: $x = [x_1, x_2, \ldots, x_n]^T$
- **Cosine similarity**: $\cos(\theta) = (u \cdot v)/(||u|| \cdot ||v||)$
- **Projection**: least squares regression

Quantum Mechanics

- **State vectors**: $|\psi\rangle$ in Hilbert space
- **Inner product**: $\langle\varphi|\psi\rangle$ (overlap of states)
- **Orthonormal basis**: $\langle e_i|e_j\rangle = \delta_{ij}$

Computer Graphics

- **3D coordinates**: position vectors
- **Normal vectors**: perpendicular to surfaces
- **Lighting**: dot product for diffuse shading

Vector Spaces

Definition

- Set V with addition and scalar multiplication
- Satisfies axioms: closure, associativity, commutativity, identity, inverses

Examples

- $\mathbb{R}^n$: n-dimensional real vectors
- $\mathbb{C}^n$: n-dimensional complex vectors
- **Polynomials**: $p(x) = a_0 + a_1x + \ldots + a_nx^n$
- **Functions**: f: $\mathbb{R} \to \mathbb{R}$

Subspaces

- **W** $\subseteq$ **V**: closed under addition and scalar multiplication
- Examples: lines through origin, planes through origin

Practice Problems

1. If $u \cdot v = 0$ and $u \cdot w = 0$, what can you say about u and span{v,w}?
2. Find the projection of v = [3,4] onto u = [1,0].
3. Why is the inner product crucial in quantum mechanics?

Key Takeaways

- Vectors have direction and magnitude
- Inner products measure alignment and enable geometry
- Orthogonal vectors are independent (perpendicular)
- Projections find closest points in subspaces
- Vector spaces generalize the concept beyond $\mathbb{R}^n$

Next Steps

Next, we'll explore Hilbert spaces—infinite-dimensional vector spaces where quantum mechanics lives.

Concepts: vector-space, inner-product, orthogonality, dimensionality

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