Lesson 3.5: Hilbert Spaces

Mathematical Intuition · Module 3 — Numbers Spaces

Node 3.5

Overview

Understanding Hilbert spaces as infinite-dimensional vector spaces with inner products, the natural setting for quantum mechanics.

Learning Objectives

- Extend vector space intuition to infinite dimensions
- Understand completeness in infinite-dimensional spaces
- Recognize Hilbert spaces in quantum mechanics and signal processing
- Build intuition for orthonormal bases and expansions

Core Concepts

What Is a Hilbert Space?

Definition

- **Vector space** with inner product
- **Complete**: all Cauchy sequences converge
- Can be finite or infinite dimensional
- Examples: $\mathbb{R}^n$, $\mathbb{C}^n$, $L^2(\mathbb{R})$, space of square-integrable functions

Key Properties

- **Inner product**: $\langle u,v \rangle$ (generalizes dot product)
- **Norm**: $||v|| = \sqrt{\langle v,v \rangle}$ (length)
- **Completeness**: limits of sequences stay in the space
- **Separable**: has countable dense subset (for most physical Hilbert spaces)

Infinite-Dimensional Spaces

Function Spaces

- $\mathbf{L^2(\mathbb{R})}$: square-integrable functions
- $\boldsymbol{\int |f(x)|^2 dx < \infty}$: finite "energy"
- Inner product: $\langle f,g \rangle = \int f^*(x)g(x)dx$

Sequence Spaces

- ℓ^2: square-summable sequences
- $\mathbf{a = (a_1, a_2, a_3, \ldots)}$ with $\Sigma_n |a_n|^2 < \infty$
- Inner product: $\langle a,b \rangle = \Sigma_n a_n{}^* b_n$

Orthonormal Bases

Finite Dimensions

- **$\{e_1, e_2, \ldots, e_n\}$**: orthonormal if $\langle e_i, e_j \rangle = \delta_{ij}$
- Any vector: $v = \Sigma_i \langle e_i, v \rangle e_i$
- Coefficients: $c_i = \langle e_i, v \rangle$

Infinite Dimensions

- **$\{e_n\}_{n=1}$**$\hat{}\infty$: countable orthonormal basis
- Expansion: $v = \Sigma_n \langle e_n, v \rangle e_n$
- Parseval's identity: $||v||^2 = \Sigma_n |\langle e_n, v \rangle|^2$

Examples of Bases

Fourier Basis

- **$e_n(x) = e\hat{}(inx)/\sqrt{}(2\pi)$**: orthonormal on $[-\pi, \pi]$
- Fourier series: $f(x) = \Sigma_n c_n e\hat{}(inx)$
- Coefficients: $c_n = \langle e_n, f \rangle = \int f(x) e\hat{}(-inx) dx/(2\pi)$

Polynomial Basis

- **Legendre polynomials**: orthogonal on [-1,1]
- **Hermite polynomials**: orthogonal with Gaussian weight
- Used in quantum harmonic oscillator

Wavelet Basis

- **Localized in time and frequency**
- Used in signal processing and compression

Quantum Mechanics in Hilbert Space

State Vectors

- $|\psi\rangle \in \mathcal{H}$: quantum state in Hilbert space
- **$||\psi|| = 1$**: normalized (total probability = 1)
- **$\langle \psi | \psi \rangle = 1$**: inner product with itself

Superposition

- **$|\psi\rangle = \Sigma_n c_n |n\rangle$**: linear combination of basis states
- **$|c_n|^2 = |\langle n | \psi \rangle|^2$**: probability of measuring state $|n\rangle$
- $\Sigma_n |c_n|^2 = 1$ (normalization)

Operators on Hilbert Space

- **$\hat{A}$:** $\mathcal{H} \to \mathcal{H}$: linear transformation
- **Hermitian operators**: $\hat{A}\dagger = \hat{A}$ (observables)
- **Unitary operators**: $\hat{U}\dagger\hat{U} = I$ (time evolution)

Spectral Theorem

- **Hermitian operator $\hat{A}$**: has real eigenvalues and orthonormal eigenvectors
- **$\hat{A} = \Sigma_n \lambda_n |n\rangle\langle n|$**: spectral decomposition
- Measurement outcomes are eigenvalues

Examples in Context

Quantum Mechanics

- **Position space**: $\psi(x) \in L^2(\mathbb{R})$
- **Momentum space**: $\varphi(p) \in L^2(\mathbb{R})$
- **Spin space**: $|\uparrow\rangle, |\downarrow\rangle \in \mathbb{C}^2$
- **Multi-qubit**: $|\psi\rangle \in \mathbb{C}\hat{}(2^n)$ (n qubits)

Signal Processing

- **Audio signals**: $f(t) \in L^2(\mathbb{R})$
- **Fourier transform**: decompose into frequencies
- **Wavelet transform**: time-frequency localization

Quantum Computing

- **Qubit state**: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$
- **n-qubit state**: $|\psi\rangle \in \mathbb{C}\hat{}(2^n)$
- **Entangled states**: cannot be written as tensor products

Functional Analysis

- **Operators**: differential, integral, multiplication
- **Eigenvalue problems**: $\hat{A}f = \lambda f$
- **Green's functions**: solutions to differential equations

Completeness and Convergence

Cauchy Sequences

- **$\{v_n\}$**: Cauchy if $||v_n - v_m|| \to 0$ as $n,m \to \infty$
- In Hilbert space: Cauchy sequences converge

Weak vs Strong Convergence

- **Strong**: $||v_n - v|| \to 0$
- **Weak**: $\langle w,v_n\rangle \to \langle w,v\rangle$ for all w
- Weak convergence is weaker condition

Practice Problems

1. Why is $L^2(\mathbb{R})$ a Hilbert space but $L^1(\mathbb{R})$ is not?
2. Show that {sin(nx), cos(nx)} forms an orthogonal set on $[0,2\pi]$.
3. What is the dimension of the Hilbert space for 10 qubits?

Key Takeaways

- Hilbert spaces are complete vector spaces with inner products
- Can be infinite-dimensional (function spaces, sequence spaces)
- Orthonormal bases allow expansion of any vector
- Quantum mechanics naturally lives in Hilbert spaces
- Completeness ensures limits of sequences stay in the space

Next Steps

Next, we'll explore operators—linear transformations that act on Hilbert spaces and represent observables in quantum mechanics.

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Concepts: hilbert-space, inner-product, convergence, dimensionality

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