# Lesson 3.6: Operators

Mathematical Intuition · Module 3 — Numbers Spaces

Node 3.6

## Overview

Understanding operators as linear transformations on vector spaces, with special focus on Hermitian and unitary operators in quantum mechanics.

## Learning Objectives

- See operators as functions that transform vectors
- Understand eigenvalues and eigenvectors geometrically
- Recognize Hermitian operators as observables
- Build intuition for unitary operators as symmetries and time evolution

## Core Concepts

### What Are Operators?

#### Definition

- **$\hat{\mathbf{A}}$: V → V**: linear transformation on vector space V
- **Linear**: $\hat{A}(\alpha u + \beta v) = \alpha\hat{A}u + \beta\hat{A}v$
- In finite dimensions: represented by matrices
- In infinite dimensions: differential operators, integral operators

#### Action on Vectors

- **$\hat{\mathbf{A}}\mathbf{v} = \mathbf{w}$**: operator transforms vector v to w
- Can change direction and magnitude
- Special directions (eigenvectors) only get scaled

### Matrix Representation

#### Finite Dimensions

- **$\mathbf{A} = [\mathbf{a}_{ij}]$**: matrix with entries $a_{ij}$
- **$\mathbf{Av} = \mathbf{w}$**: matrix-vector multiplication
- **$\mathbf{a}_{ij} = \langle \mathbf{e}_i | \hat{\mathbf{A}} | \mathbf{e}_j \rangle$**: matrix elements in basis $\{e_j\}$

#### Change of Basis

- **$\mathbf{A' = P^{-1}AP}$**: similarity transformation
- Same operator, different basis
- Eigenvalues unchanged

### Eigenvalues and Eigenvectors

### Eigenvalue Equation

- $\hat{\mathbf{A}}\mathbf{v} = \lambda\mathbf{v}$: v is eigenvector with eigenvalue $\lambda$
- v is special direction that only scales
- $\lambda$ can be real or complex

### Geometric Meaning

- Eigenvectors: directions that don't rotate under Â
- Eigenvalues: scaling factors
- Eigenspaces: span of eigenvectors with same eigenvalue

### Spectral Decomposition

- $\hat{\mathbf{A}} = \Sigma_i\lambda_i|\mathbf{v}_i\rangle\langle\mathbf{v}_i|$: sum over eigenprojectors
- Any vector: $\mathrm{v} = \Sigma_i \mathrm{c}_i|\mathrm{v}_i\rangle$
- Action: $\hat{\mathrm{A}}\mathrm{v} = \Sigma_i\lambda_i \mathrm{c}_i|\mathrm{v}_i\rangle$

### Hermitian Operators

### Definition

- $\hat{\mathbf{A}}\dagger = \hat{\mathbf{A}}$: "Hermitian" or "self-adjoint"
- $\langle\mathbf{u}|\hat{\mathbf{A}}\mathbf{v}\rangle = \langle\hat{\mathbf{A}}\mathbf{u}|\mathbf{v}\rangle$: adjoint property
- Matrix: $\mathrm{A}\dagger = (\mathrm{A}^*)^T$ (conjugate transpose)

### Properties

- **Real eigenvalues**: $\lambda_i \in \mathbb{R}$
- **Orthogonal eigenvectors**: $\langle \mathrm{v}_i|\mathrm{v}_j\rangle = 0$ if $\lambda_i \neq \lambda_j$
- **Complete basis**: eigenvectors span the space

### Physical Meaning

- **Observables in QM**: position, momentum, energy, spin
- **Measurement outcomes**: eigenvalues (always real)
- **Measurement states**: eigenvectors

### Unitary Operators

### Definition

- $\hat{\mathbf{U}}\dagger\hat{\mathbf{U}} = \hat{\mathbf{U}}\hat{\mathbf{U}}\dagger = \mathbf{I}$: "unitary"
- $\langle\hat{\mathbf{U}}\mathbf{u}|\hat{\mathbf{U}}\mathbf{v}\rangle = \langle\mathbf{u}|\mathbf{v}\rangle$: preserves inner products
- Preserves norms: $||\hat{\mathrm{U}}\mathrm{v}|| = ||\mathrm{v}||$

### Properties

- **Eigenvalues on unit circle**: $|\lambda_i| = 1$
- **Orthonormal eigenvectors**: if Hermitian
- **Reversible**: $\hat{\mathrm{U}}^{-1} = \hat{\mathrm{U}}\dagger$

### Physical Meaning

- **Time evolution**: Û(t) = e^(-iĤt/ℏ)
- **Quantum gates**: rotations on Bloch sphere
- **Symmetries**: rotations, translations, gauge transformations

**Common Operators**

**Position and Momentum**

- **x**: position operator (multiplication by x)
- **p = -i**$\hbar\partial/\partial$**x**: momentum operator
- **[x,p] = i**$\hbar$: canonical commutation relation

**Hamiltonian**

- $\hat{\mathbf{H}}$: energy operator
- $\hat{\mathbf{H}}|\mathbf{n}\rangle = \mathbf{E}_n|\mathbf{n}\rangle$: energy eigenstates
- **Time evolution**: $i\hbar\partial|\psi\rangle/\partial t = \hat{H}|\psi\rangle$

**Angular Momentum**

- **L = r** $\times$ **p**: orbital angular momentum
- $\hat{\mathbf{S}}$: spin angular momentum
- $[\mathbf{L}_i,\mathbf{L}_j] = \mathbf{i}\hbar\varepsilon_{ijk}\mathbf{L}_k$: angular momentum algebra

**Pauli Matrices**

- $\sigma_x$, $\sigma_\gamma$, $\sigma_u$: spin-1/2 operators
- $\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbf{I}$: anticommutation relations
- $[\sigma_i,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k$: commutation relations

## Examples in Context

**Quantum Mechanics**

- **Measurement**: $\hat{A}\psi = a\psi$ gives outcome a
- **Time evolution**: $|\psi(t)\rangle = e\hat{}(-i\hat{H}t/\hbar)|\psi(0)\rangle$
- **Expectation value**: $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$

**Quantum Computing**

- **Single-qubit gates**: X, Y, Z, H (Pauli and Hadamard)
- **Rotation gates**: $R_x(\theta) = e\hat{}(-i\theta\sigma_x/2)$
- **Two-qubit gates**: CNOT, SWAP, CZ

**Linear Algebra**

- **Projection**: $P^2 = P$ (idempotent)
- **Reflection**: $R^2 = I$
- **Rotation**: $R(\theta)$ in 2D or 3D

**Differential Equations**

- **Differential operators**: d/dx, $\nabla^2$, $\partial/\partial t$
- **Eigenvalue problems**: $-\nabla^2\psi = \lambda\psi$
- **Green's functions**: $(\hat{A} - \lambda I)^{-1}$

## Operator Algebra

**Commutators**

- $[\hat{\mathbf{A}},\mathbf{B}] = \hat{\mathbf{A}}\mathbf{B} - \mathbf{B}\hat{\mathbf{A}}$: measures non-commutativity
- $[\hat{\mathbf{A}},\mathbf{B}] = \mathbf{0}$: simultaneous eigenbasis exists

- **$[\hat{A}, B] \neq 0$**: uncertainty relation

**Functions of Operators**

- **$f(\hat{A}) = \Sigma_n c_n$ n**: power series
- **e^$(i\theta\hat{A})$**: exponential (rotation/evolution)
- **Spectral theorem**: $f(\hat{A}) = \Sigma_i f(\lambda_i)|v_i\rangle\langle v_i|$

**Tensor Products**

- **$\hat{A} \otimes B$**: operator on product space
- **$(\hat{A} \otimes B)(|u\rangle \otimes |v\rangle) = (\hat{A}u) \otimes (Bv)$**
- Multi-qubit operators

## Practice Problems

1. If and B commute, what can you say about their eigenvectors?
2. Why must observables be Hermitian operators?
3. Show that e^$(i\theta\sigma_x)$ is a rotation around the x-axis on the Bloch sphere.

## Key Takeaways

- Operators are linear transformations on vector spaces
- Eigenvectors are special directions that only get scaled
- Hermitian operators have real eigenvalues (observables)
- Unitary operators preserve inner products (time evolution)
- Commutators measure incompatibility of observables

## Module 3 Complete

You now understand numbers, spaces, and operators from integers to Hilbert spaces. In Module 4, we'll apply this intuition to variational quantum eigensolvers and regularization.

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**Concepts:** linear-operators, eigenvalue-decomposition, spectral-analysis, matrix-representation, unitary-evolution

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