Lesson 5.3: Spot Generators

Mathematical Intuition · Module 5 — Clarity Drills

Node 5.3

Overview

Practice recognizing operators that generate transformations: time evolution, rotations, translations, and other symmetries.

Learning Objectives

- Identify generators in exponential operators
- Understand the relationship between generators and transformations
- Recognize Lie algebras and commutation relations
- Build intuition for infinitesimal vs finite transformations

Core Concepts

What Is a Generator?

Definition

- **Generator**: Operator $\hat{G}$ that produces transformation via exponentiation
- **Transformation**: $\hat{U} = e^{i\theta\hat{G}}$ (unitary transformation)
- **Infinitesimal**: $\hat{U} \approx I + i\theta\hat{G}$ for small θ
- **Finite**: $\hat{U} = \lim_{n\to\infty}(I + i\theta\hat{G}/n)^n$

Why Exponentials?

- Exponentials turn addition into multiplication: $e^{a+b} = e^a \cdot e^b$
- Generators form Lie algebras: $[\hat{G}_i,\hat{G}_j] = i\Sigma_k f_{ijk}\hat{G}_k$
- Continuous transformations: θ parameterizes transformation
- Unitary: $e^{i\theta\hat{G}}$ is unitary if $\hat{G}$ is Hermitian

Drill Set 1: Identify the Generator

Transformation 1: $U(t) = e^{-i\hat{H}t/\hbar}$

- **Generator**: $\hat{H}$ (Hamiltonian)
- **Transformation**: Time evolution
- **Parameter**: t (time)
- **Physical meaning**: Energy generates time evolution

Transformation 2: $R(\theta) = e^{-iL_u\theta/\hbar}$

- **Generator**: L_u (angular momentum around z-axis)
- **Transformation**: Rotation around z-axis
- **Parameter**: θ (angle)
- **Physical meaning**: Angular momentum generates rotations

Transformation 3: `T(a) = e^(-iâp/ħ)`

- **Generator**: p (momentum operator)
- **Transformation**: Translation by distance a
- **Parameter**: a (displacement)
- **Physical meaning**: Momentum generates translations

Transformation 4: `B(v) = e^(-ivK/c)`

- **Generator**: K (boost generator)
- **Transformation**: Lorentz boost (special relativity)
- **Parameter**: v (velocity)
- **Physical meaning**: Boost generator produces velocity transformations

Transformation 5: `G(α) = e^(α â† - α* â)`

- **Generator**: αâ† - α*â (displacement operator)
- **Transformation**: Coherent state displacement
- **Parameter**: α (complex displacement)
- **Physical meaning**: Creates coherent states in quantum optics

Drill Set 2: Commutators and Algebras

Commutator 1: `[x,p] = iħ`

- **Generators**: x (position), p (momentum)
- **Algebra**: Heisenberg algebra
- **Transformations**: Translations in momentum and position space
- **Physical meaning**: Canonical commutation relation

Commutator 2: `[L_i,L_j] = iħε_ijk L_k`

- **Generators**: L_x, L_γ, L_u (angular momentum components)
- **Algebra**: SO(3) (rotation group)
- **Transformations**: Rotations in 3D space
- **Physical meaning**: Angular momentum generates rotations

Commutator 3: `[σ_i,σ_j] = 2iε_ijk σ_k`

- **Generators**: σ_x, σ_γ, σ_u (Pauli matrices)
- **Algebra**: SU(2) (spin group)
- **Transformations**: Rotations on Bloch sphere
- **Physical meaning**: Spin operators generate qubit rotations

Commutator 4: `[â,â†] = 1`

- **Generators**: â (annihilation), â† (creation)
- **Algebra**: Harmonic oscillator algebra
- **Transformations**: Ladder operators (change energy level)
- **Physical meaning**: Create/destroy quanta

Commutator 5: `[Ĥ,Ũ(t)] = 0` (if $\hat{H}$ time-independent)

- **Generators**: $\hat{H}$ (Hamiltonian), $\hat{U}$(t) (time evolution)
- **Meaning**: Hamiltonian commutes with its own time evolution
- **Consequence**: Energy is conserved

Drill Set 3: Recognize Transformation Type

For each exponential, identify what transformation it generates:

Transformation 6: `e^(-iθσ_x/2)`

- **Type**: Rotation around x-axis on Bloch sphere
- **Generator**: σ_x (Pauli X)
- **Angle**: θ
- **Quantum gate**: $R_x(\theta)$

Transformation 7: `e^(-iθσ_γ/2)`

- **Type**: Rotation around y-axis on Bloch sphere
- **Generator**: σ_γ (Pauli Y)
- **Angle**: θ
- **Quantum gate**: $R_\gamma(\theta)$

Transformation 8: `e^(-iθσ_u/2)`

- **Type**: Rotation around z-axis on Bloch sphere (phase)
- **Generator**: σ_u (Pauli Z)
- **Angle**: θ
- **Quantum gate**: $R_u(\theta)$

Transformation 9: `e^(-iθ(σ_x⊗σ_x + σ_γ⊗σ_γ)/2)`

- **Type**: Two-qubit entangling gate
- **Generator**: $\sigma_x \otimes \sigma_x + \sigma_\gamma \otimes \sigma_\gamma$ (XX + YY interaction)
- **Angle**: θ
- **Effect**: Creates entanglement

Transformation 10: `e^(-βĤ)`

- **Type**: Thermal state / Gibbs state
- **Generator**: $\hat{H}$ (Hamiltonian)
- **Parameter**: $\beta = 1/(k\_BT)$ (inverse temperature)
- **Physical meaning**: Boltzmann distribution

Drill Set 4: Infinitesimal Transformations

Problem 1

Given: `U(ε) = e^(-iεĜ) ≈ I - iεĜ` for small ε

Question: What is the infinitesimal change in state $|\psi\rangle$?

Answer: `|ψ'⟩ = U(ε)|ψ⟩ ≈ (I - iεĜ)|ψ⟩ = |ψ⟩ - iεĜ|ψ⟩` - Change: `-iεĜ|ψ⟩` - First-order in ε

Problem 2

Given: Time evolution `|ψ(t+dt)⟩ = e^(-iĤdt/ħ)|ψ(t)⟩`

Question: What is the time derivative?

Answer: `d|ψ⟩/dt = lim(dt→0)[|ψ(t+dt)⟩ - |ψ(t)⟩]/dt = -iĤ|ψ⟩/ħ` - Schrödinger equation - $\hat{H}$ generates time evolution

Problem 3

Given: Rotation $\texttt{R(d}\theta\texttt{) = e^(-iL}_u\texttt{d}\theta/\hbar\texttt{)} \approx \texttt{I - iL}_u\texttt{d}\theta/\hbar$

Question: How does a state change under infinitesimal rotation?

Answer: $\texttt{|}\psi\texttt{'}\rangle \approx \texttt{|}\psi\rangle\ \texttt{- iL}_u\texttt{d}\theta/\hbar\texttt{|}\psi\rangle$ - Change proportional to angular momentum - dθ is infinitesimal angle

Drill Set 5: Symmetries and Conservation

Symmetry 1: Time Translation

- **Generator**: $\hat{\text{H}}$ (Hamiltonian)
- **Transformation**: U(t) = e^(-i$\hat{\text{H}}$t/$\hbar$)
- **Conserved quantity**: Energy
- **Noether's theorem**: Time translation symmetry → energy conservation

Symmetry 2: Spatial Translation

- **Generator**: p (momentum)
- **Transformation**: T(a) = e^(-iâp/$\hbar$)
- **Conserved quantity**: Momentum
- **Noether's theorem**: Spatial translation symmetry → momentum conservation

Symmetry 3: Rotation

- **Generator**: L (angular momentum)
- **Transformation**: R(θ) = e^(-iθL/$\hbar$)
- **Conserved quantity**: Angular momentum
- **Noether's theorem**: Rotational symmetry → angular momentum conservation

Symmetry 4: Gauge Transformation

- **Generator**: Q (charge operator)
- **Transformation**: G(α) = e^(iαQ)
- **Conserved quantity**: Charge
- **Noether's theorem**: Gauge symmetry → charge conservation

Practice Exercises

Identify the Generator

1. $\texttt{U = e^(-i}\theta\hat{\text{J}}_x\texttt{)}$ (What does $\hat{\text{J}}_x$ generate?)
2. `T = e^(ikx)` (What does x generate?)
3. $\texttt{D(}\alpha\texttt{) = e^(}\alpha\hat{\text{a}}\texttt{† - }\alpha\texttt{*}\hat{\text{a}}\texttt{)}$ (What transformation?)
4. $\texttt{S(r) = e^(r(}\hat{\text{a}}\texttt{†}^2\texttt{ - }\hat{\text{a}}^2\texttt{)/2)}$ (What is this? Hint: squeezing)
5. $\texttt{P = e^(i}\pi\hat{\text{a}}\texttt{†}\hat{\text{a}}\texttt{)}$ (What does this do? Hint: parity)

Compute Infinitesimal Transformations

6. If $\texttt{U(}\varepsilon\texttt{) = e^(-i}\varepsilon\sigma_x\texttt{)}$, what is $\texttt{U(}\varepsilon\texttt{)|0}\rangle$ to first order in ε?
7. If $\texttt{R(d}\theta\texttt{) = e^(-iL}_u\texttt{d}\theta\texttt{)}$, what is $\texttt{R(d}\theta\texttt{)|}\ell\texttt{,m}\rangle$ to first order?
8. If `T(dx) = e^(-ipdx/`$\hbar$`)`, what is $\texttt{T(dx)}\psi\texttt{(x)}$ to first order?

Recognize Algebras

9. Given $[\hat{J}_i, \hat{J}_j] = i\varepsilon_{ijk}\hat{J}_k$, what group does this represent?
10. Given $\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$, what does this anticommutation relation tell you?

Key Takeaways

- Generators produce transformations via exponentiation: $U = e^{(i\theta\hat{G})}$
- Hermitian generators $\rightarrow$ unitary transformations
- Commutators define Lie algebras and determine transformation structure
- Infinitesimal transformations: $U \approx I + i\theta\hat{G}$
- Symmetries and conservation laws: generator = conserved quantity (Noether)

Next Steps

Next, we'll practice understanding how parameters scale, shift, and warp functions and spaces.

Mathematical Intuition · Module 5 — Clarity Drills · Node 5.3
Concepts: unitary-evolution, dynamical-systems, linear-operators

hbar.university — own.your.cognition