Lesson 5.4: Scaling, Shifting, Warping

Mathematical Intuition · Module 5 — Clarity Drills

Node 5.4

Overview

Practice understanding how parameters scale, shift, and warp functions and spaces.

Learning Objectives

- Recognize scaling transformations (multiplication)
- Identify shifting transformations (addition)
- Understand warping (nonlinear transformations)
- Build intuition for how parameters affect function behavior

Core Concepts

Three Fundamental Transformations

Scaling

- **Vertical**: `f(x)` → `A·f(x)` (multiply output)
- **Horizontal**: `f(x)` → `f(x/a)` (divide input)
- **Effect**: Changes amplitude or frequency

Shifting

- **Vertical**: `f(x)` → `f(x) + b` (add to output)
- **Horizontal**: `f(x)` → `f(x - c)` (subtract from input)
- **Effect**: Moves function up/down or left/right

Warping

- **Nonlinear**: `f(x)` → `f(g(x))` (compose with nonlinear g)
- **Effect**: Distorts function shape
- **Examples**: exponential, logarithmic, power transformations

Drill Set 1: Identify the Transformation

Function 1: `y = 3sin(x)`

- **Transformation**: Vertical scaling
- **Parameter**: 3 (amplitude)
- **Effect**: Oscillates between -3 and 3 (instead of -1 and 1)

Function 2: `y = sin(2x)`

- **Transformation**: Horizontal scaling
- **Parameter**: 2 (frequency multiplier)

- **Effect**: Oscillates twice as fast (period = π instead of 2π)

Function 3: `y = sin(x) + 2`

- **Transformation**: Vertical shift
- **Parameter**: 2 (offset)
- **Effect**: Oscillates between 1 and 3 (shifted up by 2)

Function 4: `y = sin(x - π/2)`

- **Transformation**: Horizontal shift (phase shift)
- **Parameter**: $\pi/2$ (phase)
- **Effect**: Shifted right by $\pi/2$ (sin becomes cos)

Function 5: `y = e^(2x)`

- **Transformation**: Horizontal scaling (warping)
- **Parameter**: 2 (growth rate)
- **Effect**: Grows twice as fast as e^x

Drill Set 2: General Form Analysis

General Sinusoid: `y = A·sin(ωx + φ) + C`

- **A**: Amplitude (vertical scaling)
- ω: Angular frequency (horizontal scaling)
- φ: Phase shift (horizontal shift)
- **C**: Vertical offset (vertical shift)

General Gaussian: `y = A·e^(-(x-μ)²/2σ²)`

- **A**: Amplitude (vertical scaling)
- μ: Mean (horizontal shift)
- σ: Standard deviation (horizontal scaling)
- **Effect**: Bell curve centered at μ with width σ

General Exponential: `y = A·e^(λx) + C`

- **A**: Initial amplitude (vertical scaling)
- λ: Growth/decay rate (warping)
- **C**: Asymptote (vertical shift)

General Power Law: `y = A·x^n`

- **A**: Coefficient (vertical scaling)
- **n**: Exponent (warping)
- **Effect**: n > 1 (superlinear), n < 1 (sublinear), n < 0 (decay)

Drill Set 3: Parameter Effects

Problem 1: `f(x) = A·sin(x)`

Question: What happens as A increases?

Answer: - Amplitude increases - Oscillates between -A and A - Frequency unchanged - Period unchanged

Problem 2: $f(x) = \sin(\omega x)$

Question: What happens as ω increases?

Answer: - Frequency increases - Oscillates faster - Period decreases: $T = 2\pi/\omega$ - Amplitude unchanged

Problem 3: $f(x) = e^{(-x/\tau)}$

Question: What happens as τ increases?

Answer: - Decay slows down - τ is time constant (time to decay to 1/e) - Larger $\tau \rightarrow$ longer decay - Horizontal scaling

Problem 4: $f(x) = (x - \mu)^2$

Question: What happens as μ changes?

Answer: - Parabola shifts horizontally - Vertex moves to $x = \mu$ - Shape unchanged - Horizontal shift

Problem 5: $f(x) = 1/(1 + e^{(-k(x-x_0))})$

Question: What happens as k increases?

Answer: - Sigmoid becomes steeper - Transition at $x = x_0$ becomes sharper - k controls "temperature" (sharpness) - Warping transformation

Drill Set 4: Quantum Mechanics Examples

Quantum Harmonic Oscillator: $\psi_n(x) = A_n \cdot H_n(x/a) \cdot e^{(-x^2/2a^2)}$

- $\mathbf{A}_n$: Normalization (vertical scaling)
- $\mathbf{H}_n$: Hermite polynomial (shape)
- $\mathbf{a} = \sqrt{(\hbar/m\omega)}$: Length scale (horizontal scaling)
- **Effect**: a determines spatial extent of wavefunction

Gaussian Wave Packet: $\psi(x,t) = A \cdot e^{(-(x-vt)^2/2\sigma^2)} \cdot e^{(i(kx-\omega t))}$

- **A**: Amplitude
- **vt**: Center position (horizontal shift, moving)
- σ: Width (horizontal scaling)
- **k,** ω: Wave vector and frequency (phase warping)

Coherent State: $|\alpha\rangle = e^{(-|\alpha|^2/2)} \Sigma_n (\alpha^n/\sqrt{(n!)})|n\rangle$

- α: Complex displacement parameter
- $|\alpha|$: Amplitude (scaling)
- **arg**(α): Phase (rotation)
- **Effect**: Displaced harmonic oscillator ground state

Drill Set 5: Machine Learning Examples

Sigmoid Activation: $\sigma(x) = 1/(1 + e^{(-x)})$

- **Scaling**: $\sigma(kx)$ makes transition steeper ($k > 1$) or gentler ($k < 1$)
- **Shifting**: $\sigma(x - b)$ moves transition point to $x = b$
- **Effect**: Controls neuron activation threshold and sharpness

ReLU Activation: `ReLU(x) = max(0, x)`

- **Scaling**: `ReLU(kx)` scales positive outputs by k
- **Shifting**: `ReLU(x - b)` creates threshold at $x = b$
- **Leaky ReLU**: $\texttt{max(}\alpha\texttt{x, x)}$ allows small negative slope α

Softmax: $\texttt{P(y=k) = e\^{}(z}_k\texttt{)/}\Sigma_j\texttt{e\^{}(z}_j\texttt{)}$

- **Temperature scaling**: $\texttt{P(y=k) = e\^{}(z}_k\texttt{/T)/}\Sigma_j\texttt{e\^{}(z}_j\texttt{/T)}$
- $\mathbf{T > 1}$: Softer probabilities (more uniform)
- $\mathbf{T < 1}$: Sharper probabilities (more peaked)
- $\mathbf{T \to 0}$: Approaches argmax (one-hot)

Drill Set 6: Identify Parameter Roles

For each function, identify which parameters scale, shift, or warp:

Function 6: $\texttt{y = A}\cdot\texttt{cos(}\omega\texttt{t + }\varphi\texttt{) + C}$

- **A**: Vertical scaling (amplitude)
- ω: Horizontal scaling (frequency)
- φ: Horizontal shift (phase)
- **C**: Vertical shift (offset)

Function 7: $\texttt{y = A/(1 + (x/x}_0\texttt{)\^{}n)}$

- **A**: Vertical scaling (maximum value)
- $\mathbf{x_0}$: Horizontal scaling (half-max point)
- **n**: Warping (steepness of transition)

Function 8: $\psi\texttt{(x) = }\sqrt{}\texttt{(2/L)}\cdot\texttt{sin(n}\pi\texttt{x/L)}$

- **L**: Horizontal scaling (box size)
- **n**: Warping (number of nodes, energy level)
- $\sqrt{}\mathbf{(2/L)}$: Vertical scaling (normalization)

Function 9: $\texttt{E(k) = E}_0\texttt{ + }\hbar^2\texttt{k}^2\texttt{/2m}$

- $\mathbf{E_0}$: Vertical shift (band minimum)
- $\hbar^2\mathbf{/2m}$: Vertical scaling (effective mass)
- $\mathbf{k^2}$: Warping (quadratic dispersion)

Function 10: $\texttt{P(x|}\mu,\sigma\texttt{) = (1/}\sqrt{}\texttt{(2}\pi\sigma^2\texttt{))}\cdot\texttt{e\^{}(-(x-}\mu\texttt{)}^2\texttt{/2}\sigma^2\texttt{)}$

- μ: Horizontal shift (mean)
- σ: Horizontal scaling (standard deviation)
- $\mathbf{1/}\sqrt{}\mathbf{(2\pi\sigma^2)}$: Vertical scaling (normalization)

Practice Exercises

Identify Transformations

1. $\texttt{f(x) = 5}\cdot\texttt{e\^{}(3x) + 2}$
2. $\texttt{g(t) = 10}\cdot\texttt{sin(2}\pi\texttt{t/T - }\pi\texttt{/4)}$
3. $\texttt{h(x) = (x - 3)}^2\texttt{ + 1}$
4. $\texttt{p(x) = (1/}\sigma\sqrt{}\texttt{(2}\pi\texttt{))}\cdot\texttt{e\^{}(-(x-}\mu\texttt{)}^2\texttt{/2}\sigma^2\texttt{)}$
5. $\texttt{E(}\theta\texttt{) = E}_0\cdot\texttt{cos}^2\texttt{(}\theta\texttt{/2)}$

Predict Effects

6. If $f(x) = \sin(x)$, what is $f(2x) + 3$?
7. If $g(x) = e^{(-x)}$, what is $2 \cdot g(x/3)$?
8. If $h(x) = x^2$, what is $h(x - 2) + 5$?

Reverse Engineering

9. Given $y = 3 \cdot \sin(4x - \pi) + 2$, identify all transformations from $\sin(x)$.
10. Given $\psi(x) = (1/\sqrt{a}) \cdot e^{(-x^2/2a^2)}$, what happens as $a \to 0$? As $a \to \infty$?

Key Takeaways

- Scaling: multiply (vertical) or divide input (horizontal)
- Shifting: add (vertical) or subtract from input (horizontal)
- Warping: nonlinear transformations change function shape
- Parameters control amplitude, frequency, phase, width, center
- Understanding parameter effects is crucial for tuning models and interpreting equations

Next Steps

Next, we'll practice reverse engineering equations—given an equation, deduce what process it describes.

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Concepts: linear-operators

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