Lesson 6.4: Maxwell's Equations

Mathematical Intuition · Module 6 — Capstone

Node 6.4

Overview

Complete decoding of Maxwell's equations—the four fundamental equations that describe all of classical electromagnetism.

The Four Equations

Differential Form

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1. ∇·E = ρ/ε₀            (Gauss's Law)
2. ∇·B = 0               (No magnetic monopoles)
3. ∇×E = -∂B/∂t           (Faraday's Law)
4. ∇×B = μ₀J + μ₀ε₀∂E/∂t  (Ampère-Maxwell Law)
```

Integral Form

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1. ∮ E·dA = Q_enc/ε₀
2. ∮ B·dA = 0
3. ∮ E·dl = -dΦ_B/dt
4. ∮ B·dl = μ₀I_enc + μ₀ε₀dΦ_E/dt
```

Symbol-by-Symbol Breakdown

Common Symbols

E (electric field) - **Vector field**: E(r,t) = [E_x, E_γ, E_u] - **Units**: V/m (volts per meter) or N/C (newtons per coulomb) - **Physical meaning**: Force per unit charge - **Source**: Charges (static and moving)

B (magnetic field) - **Vector field**: B(r,t) = [B_x, B_γ, B_u] - **Units**: T (tesla) or Wb/m^2 (weber per square meter) - **Physical meaning**: Force on moving charges - **Source**: Moving charges (currents)

ρ (charge density) - **Scalar field**: ρ(r,t) - **Units**: C/m^3 (coulombs per cubic meter) - **Physical meaning**: Charge per unit volume - **Relation to charge**: $Q = \int \rho \, dV$

J (current density) - **Vector field**: J(r,t) = [J_x, J_γ, J_u] - **Units**: A/m^2 (amperes per square meter) - **Physical meaning**: Current per unit area - **Relation to current**: $I = \int J \cdot dA$

ε_0 (permittivity of free space) - **Value**: 8.854×10^{-12} F/m (farads per meter) - **Physical meaning**: How much electric field is produced by charges - **Role**: Couples charges to electric field

μ_0 (permeability of free space) - **Value**: $4\pi \times 10^{-7}$ H/m (henries per meter) - **Physical meaning**: How much magnetic field is produced by currents - **Role**: Couples currents to magnetic field

c (speed of light) - **Value**: $c = 1/\sqrt{}(\mu_0\varepsilon_0) \approx 3 \times 10^8$ m/s - **Emerges from**: Maxwell's equations - **Significance**: Electromagnetic waves travel at c

Equation 1: Gauss's Law

Differential Form: $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$

∇ · E (divergence of E) - **Definition**: $\nabla \cdot E = \partial E_x/\partial x + \partial E_\gamma/\partial y + \partial E_u/\partial z$ - **Meaning**: "Outflow" of electric field from a point - **Positive**: Field lines emanate (source) - **Negative**: Field lines converge (sink)

ρ/ε_0 (charge density scaled) - **Meaning**: Charge density creates divergence - **Positive** ρ: Electric field points outward - **Negative** ρ: Electric field points inward

Physical interpretation: - "Electric field lines begin and end on charges" - Positive charges are sources (field lines out) - Negative charges are sinks (field lines in) - No charges → no divergence (field lines continuous)

Integral Form: $\oint$ E·dA = Q_enc/ε_0

∮ E · dA (flux through closed surface) - **E · dA**: Electric field dot surface element - $\oint$: Integral over closed surface - **Meaning**: Total electric field passing through surface

Q_enc (enclosed charge) - **Definition**: Q_enc = ∫_V ρ dV - **Meaning**: Total charge inside surface

Physical interpretation: - "Total flux through surface equals enclosed charge" - Gauss's law for calculating E from symmetric charge distributions - Divergence theorem connects differential and integral forms

Equation 2: No Magnetic Monopoles

Differential Form: $\nabla \cdot \mathbf{B} = 0$

∇ · B (divergence of B) - **Always zero**: No magnetic charges - **Meaning**: Magnetic field lines never begin or end - **Consequence**: Magnetic field lines form closed loops

Physical interpretation: - "No magnetic monopoles exist" - Magnetic field lines always form closed loops - Every north pole has a south pole - Fundamental asymmetry between E and B

Integral Form: $\oint$ B·dA = 0

∮ B · dA (magnetic flux through closed surface) - **Always zero**: What goes in must come out - **Meaning**: No net magnetic flux through any closed surface

Physical interpretation: - "Magnetic field lines cannot begin or end" - If field lines enter surface, they must exit - No isolated magnetic charges

Equation 3: Faraday's Law

Differential Form: $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

∇×E (curl of E) - **Definition**: $\nabla \times E = [\partial E_u/\partial y - \partial E_\gamma/\partial z, \partial E_x/\partial z - \partial E_u/\partial x, \partial E_\gamma/\partial x - \partial E_x/\partial y]$ - **Meaning**: "Circulation" or "rotation" of electric field - **Non-zero**: Electric field has vortex structure

-∂B/∂t (negative time derivative of B) - **Meaning**: Changing magnetic field - **Negative sign**: Lenz's law (opposes change)

Physical interpretation: - "Changing magnetic field creates circulating electric field" - Basis of electromagnetic induction - Generators, transformers, inductors

Integral Form: $\oint$ E·dl = -dΦ_B/dt

∮ E · dl (line integral of E around closed loop) - **Meaning**: Electromotive force (EMF) around loop - **Units**: Volts

Φ_B (magnetic flux) - **Definition**: Φ_B = ∫B · dA - **Units**: Wb (webers) - **Meaning**: Total magnetic field through loop

-dΦ_B/dt (rate of change of flux) - **Meaning**: How fast magnetic flux is changing - **Negative sign**: Induced EMF opposes change (Lenz's law)

Physical interpretation: - "Changing magnetic flux induces EMF" - Faraday's law of induction - Moving magnet through coil → current

Equation 4: Ampère-Maxwell Law

Differential Form: $\nabla\times B = \mu_0 J + \mu_0\varepsilon_0\partial E/\partial t$

$\nabla\times$**B** (curl of B) - **Meaning**: Circulation of magnetic field - **Sources**: Currents and changing electric fields

μ_0**J** (current term) - **Ampère's law**: Currents create circulating magnetic field - **Right-hand rule**: Curl fingers with current, thumb points along B

$\mu_0\varepsilon_0\partial$**E**/$\partial$**t** (displacement current) - **Maxwell's addition**: Changing electric field acts like current - **Crucial**: Enables electromagnetic waves - **Symmetry**: Completes parallel with Faraday's law

Physical interpretation: - "Magnetic field circulates around currents and changing electric fields" - Currents create magnetic fields (Ampère) - Changing E creates magnetic fields (Maxwell's insight) - Displacement current enables wave propagation

Integral Form: $\oint$ B·dl = μ_0I_enc + $\mu_0\varepsilon_0$dΦ_E/dt

$\oint$ **B · dl** (line integral of B around closed loop) - **Meaning**: Circulation of magnetic field - **Related to**: Current enclosed by loop

μ_0**I_enc** (enclosed current) - **Definition**: I_enc = $\int$J · dA - **Meaning**: Total current through surface bounded by loop

$\mu_0\varepsilon_0$**dΦ_E/dt** (displacement current term) - **Φ_E**: Electric flux through surface - **dΦ_E/dt**: Rate of change of electric flux - **Acts like current**: Even without physical charges moving

Physical interpretation: - "Magnetic field circulates around currents" - Ampère's law for calculating B from currents - Displacement current: changing E field equivalent to current

Electromagnetic Waves

Wave Equation from Maxwell

Taking curl of Faraday's law and using Ampère-Maxwell:

$$\nabla^2 E - \mu_0\varepsilon_0\partial^2 E/\partial t^2 = 0$$
$$\nabla^2 B - \mu_0\varepsilon_0\partial^2 B/\partial t^2 = 0$$

Wave speed:

$$c = 1/\sqrt{(\mu_0\varepsilon_0)} \approx 3 \times 10^8 \text{ m/s}$$

Physical meaning: - E and B satisfy wave equation - Waves propagate at speed c (light!) - E and B oscillate perpendicular to each other and to propagation direction

Plane Wave Solution

$$E = E_0 \sin(k\cdot r - \omega t)$$
$$B = B_0 \sin(k\cdot r - \omega t)$$

Properties: - **E** $\perp$ **B**: Electric and magnetic fields perpendicular - **E** $\perp$ **k, B** $\perp$ **k**: Both perpendicular to propagation direction - **|E| = c|B|**: Magnitudes related by c - ω **= ck**: Dispersion relation

Energy and Momentum

Energy Density

$u = ½(\varepsilon_0 E^2 + B^2/\mu_0)$

- Electric energy: $½\varepsilon_0 E^2$
- Magnetic energy: $½B^2/\mu_0$

Poynting Vector (Energy Flux)

$S = (1/\mu_0) E \times B$

- **Meaning**: Energy flow per unit area
- **Direction**: E×B (perpendicular to both)
- **Units**: W/m^2 (watts per square meter)

Momentum Density

$g = \varepsilon_0 E \times B = S/c^2$

- Electromagnetic fields carry momentum
- Radiation pressure

Conservation Laws

Charge Conservation

$\partial\rho/\partial t + \nabla \cdot J = 0$

- **Meaning**: Charge is conserved
- **Derived from**: Maxwell's equations (take divergence of Ampère-Maxwell)

Energy Conservation (Poynting's Theorem)

$\partial u/\partial t + \nabla \cdot S = -J \cdot E$

- **u**: Energy density
- **S**: Poynting vector (energy flux)
- **-J · E**: Work done by field on charges

Applications

Static Fields ($\partial/\partial t = 0$)

Electrostatics: - $\nabla \cdot E = \rho/\varepsilon_0$ (Gauss) - $\nabla \times E = 0$ (no curl → $E = -\nabla\varphi$)

Magnetostatics: - $\nabla \cdot B = 0$ (no monopoles) - $\nabla \times B = \mu_0 J$ (Ampère)

Electromagnetic Induction

- Generators: Rotating coil in magnetic field
- Transformers: Changing current in primary → induced EMF in secondary
- Inductors: Changing current → self-induced EMF

Electromagnetic Waves

- Radio waves, microwaves, infrared, visible light, UV, X-rays, gamma rays
- All described by Maxwell's equations
- Different frequencies, same physics

Practice Problems

1. **Gauss's law**: Calculate E field of point charge Q using integral form.
2. **Faraday's law**: A circular loop of radius r is in a uniform B field increasing at rate dB/dt. Find induced EMF.
3. **Ampère's law**: Calculate B field inside a long straight wire carrying current I.
4. **Wave equation**: Derive $\nabla^2 E = \mu_0 \varepsilon_0 \partial^2 E/\partial t^2$ from Maxwell's equations.
5. **Poynting vector**: For a plane wave with $E = E_0 \sin(kz\text{-}\omega t)x$, find S.

Key Takeaways

- Four equations describe all classical electromagnetism
- Gauss: Charges create electric field (divergence)
- No monopoles: Magnetic field lines form closed loops
- Faraday: Changing B creates circulating E (induction)
- Ampère-Maxwell: Currents and changing E create circulating B
- Together: Predict electromagnetic waves at speed c
- Unified electricity, magnetism, and light

Historical Note

James Clerk Maxwell (1865): Published unified theory - Combined known laws (Gauss, Faraday, Ampère) - Added displacement current term ($\mu_0 \varepsilon_0 \partial E/\partial t$) - Predicted electromagnetic waves - **Heinrich Hertz** (1887): Experimentally confirmed waves - Foundation of modern physics and technology

Next Steps

Next, we'll decode logistic regression—the fundamental equation of binary classification in machine learning.

Mathematical Intuition · Module 6 — Capstone · Node 6.4
Concepts: dynamical-systems, vector-space

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