Lesson 6.6: Fourier Transform

Mathematical Intuition · Module 6 — Capstone

Node 6.6

Overview

Complete decoding of the Fourier transform—the fundamental tool that decomposes signals into frequency components.

The Core Equations

Fourier Transform (Continuous)

`F(ω) = ∫_{-∞}^{∞} f(t)e^{-iωt} dt`

Inverse Fourier Transform

`f(t) = 1/(2π) ∫_{-∞}^{∞} F(ω)e^{iωt} dω`

Discrete Fourier Transform (DFT)

`F_k = Σ_{n=0}^{N-1} f_n e^{-2πikn/N}`

Symbol-by-Symbol Breakdown

Time and Frequency Domains

f(t) (time-domain signal) - **Domain**: $t \in \mathbb{R}$ (time) - **Values**: Real or complex - **Meaning**: Signal as function of time - **Examples**: Audio waveform, voltage over time, stock price

F(ω) (frequency-domain representation) - **Domain**: $\omega \in \mathbb{R}$ (angular frequency) - **Values**: Complex numbers - **Meaning**: Amplitude and phase at each frequency - **Interpretation**: "How much of frequency ω is in f(t)?"

ω (angular frequency) - **Units**: rad/s (radians per second) - **Relation to frequency**: $\omega = 2\pi$f (f in Hz) - **Meaning**: Rate of oscillation

The Transform: `F(ω) = ∫_{-∞}^{∞} f(t)e^{-iωt} dt`

∫_{-∞}^{∞} . . . dt (integral over all time) - **Meaning**: Sum up contributions from all time points - **Accumulation**: Weighted sum of signal

f(t) (signal at time t) - **Multiplied by**: Complex exponential - **Contribution**: Each time point contributes to each frequency

e^{-iωt} (complex exponential) - **Euler's formula**: e^{-iωt} = cos(ωt) - i sin(ωt) - **Meaning**: Basis function (pure frequency ω) - **Real part**: cos(ωt) (cosine wave) - **Imaginary part**: -sin(ωt) (sine wave)

Why complex exponentials? - Encode both amplitude and phase - Orthogonal basis functions - Simplify convolution (multiplication in frequency domain) - Eigenfunctions of LTI systems

Complete Interpretation

$F(\omega) = \int f(t)e^{-i\omega t}\, dt$

Plain language: "The Fourier transform at frequency ω is the overlap between the signal f(t) and a pure oscillation at frequency ω."

Process: 1. Take signal f(t) 2. Multiply by $e^{-i\omega t}$ (oscillation at frequency ω) 3. Integrate over all time 4. Result: Complex number $F(\omega)$

$F(\omega)$ as complex number:

$F(\omega) = |F(\omega)|e^{i\varphi(\omega)}$

- **$|F(\omega)|$**: Magnitude (amplitude at frequency ω)
- **$\varphi(\omega)$**: Phase (time shift of that frequency component)

Inverse Transform: $f(t) = 1/(2\pi) \int F(\omega)e^{i\omega t}\, d\omega$

Meaning: "Reconstruct signal by summing up all frequency components"

Process: 1. Take frequency component $F(\omega)$ 2. Multiply by $e^{i\omega t}$ (oscillation at frequency ω) 3. Integrate over all frequencies 4. Scale by $1/(2\pi)$ 5. Result: Original signal f(t)

Interpretation: Every signal is a superposition of pure frequencies

$1/(2\pi)$ (normalization factor) - Ensures $f(t) = \mathcal{F}^{-1}[\mathcal{F}[f(t)]]$ - Convention: Can be split between forward and inverse - Alternative: $1/\sqrt{(2\pi)}$ on both

Properties of Fourier Transform

Linearity

$\mathcal{F}[af(t) + bg(t)] = aF(\omega) + bG(\omega)$

- Transform of sum = sum of transforms
- Scaling preserved

Time Shift

$\mathcal{F}[f(t - t_0)] = e^{-i\omega t_0}F(\omega)$

- Shifting in time → phase shift in frequency
- Magnitude unchanged: $|F(\omega)|$ same

Frequency Shift

$\mathcal{F}[e^{i\omega_0 t}f(t)] = F(\omega - \omega_0)$

- Modulation in time → shift in frequency
- Basis of AM/FM radio

Scaling

$\mathcal{F}[f(at)] = 1/|a|\ F(\omega/a)$

- Compress time → stretch frequency
- Stretch time → compress frequency
- Uncertainty principle manifestation

Convolution Theorem

$\mathcal{F}$[f * g] = F(ω)·G(ω)

- Convolution in time → multiplication in frequency
- Crucial for signal processing
- Filtering = multiplication in frequency domain

Parseval's Theorem (Energy Conservation)

$\int$|f(t)|^2 dt = 1/(2π) $\int$|F(ω)|^2 dω

- Total energy same in time and frequency domains
- Energy preserved under transform

Common Transform Pairs

Delta Function

f(t) = δ(t) $\longleftrightarrow$ F(ω) = 1

- Impulse at t=0 → all frequencies equally
- Flat spectrum

Constant

f(t) = 1 $\longleftrightarrow$ F(ω) = 2$\pi\delta$(ω)

- DC signal → spike at ω=0
- No oscillation → zero frequency

Cosine

f(t) = cos(ω_0t) $\longleftrightarrow$ F(ω) = π[δ(ω-ω_0) + δ(ω+ω_0)]

- Pure frequency → two spikes ($\pm\omega_0$)
- Positive and negative frequencies

Gaussian

f(t) = e^{-t^2/2σ^2} $\longleftrightarrow$ F(ω) = $\sigma\surd$(2π)e^{-$\sigma^2\omega^2$/2}

- Gaussian → Gaussian
- Width inversely related: narrow in time → wide in frequency

Rectangular Pulse

f(t) = rect(t/T) $\longleftrightarrow$ F(ω) = T·sinc(ωT/2)

- Box function → sinc function
- Sharp edges → high frequencies

Discrete Fourier Transform (DFT)

Definition: F_k = Σ_{n=0}^{N-1} f_n e^{-2πikn/N}

f__n (discrete time samples) - **Index**: n = 0, 1, 2, ..., N-1 - **Meaning**: Signal sampled at N points - **Spacing**: Δt (sampling interval)

F__k (discrete frequency bins) - **Index**: k = 0, 1, 2, ..., N-1 - **Meaning**: Frequency components - **Frequency**: ω_k = 2πk/(NΔt)

$e^{-2\pi ikn/N}$ (DFT basis) - **Discrete version** of $e^{-i\omega t}$ - **Periodic**: $e^{-2\pi i(k+N)n/N} = e^{-2\pi ikn/N}$ - **Orthogonal**: Different k values are orthogonal

Fast Fourier Transform (FFT)

Naive DFT: $O(N^2)$ operations **FFT algorithm**: O(N log N) operations - **Cooley-Tukey algorithm** (1965) - Exploits symmetry and periodicity - Revolutionized signal processing

Applications

Signal Processing

Audio compression (MP3, AAC): 1. Transform audio to frequency domain 2. Remove inaudible frequencies 3. Quantize remaining frequencies 4. Inverse transform

Noise filtering: 1. Transform noisy signal 2. Identify noise frequencies 3. Attenuate or remove 4. Inverse transform

Image Processing

2D Fourier Transform:

$$F(u,v) = \iint f(x,y)e^{-i(ux+vy)}\, dx\, dy$$

- Decomposes image into spatial frequencies
- JPEG compression uses DCT (related to Fourier)

Quantum Mechanics

Position ↔ Momentum:

$$\psi(p) = 1/\sqrt{(2\pi\hbar)} \int \psi(x)e^{-ipx/\hbar}\, dx$$

- Fourier transform relates position and momentum representations
- Uncertainty principle: $\Delta x\Delta p \geq \hbar/2$

Differential Equations

Solving PDEs: - Transform equation to frequency domain - Differential operators → algebraic - Solve algebraically - Inverse transform

Example (heat equation):

$$\partial u/\partial t = \alpha\nabla^2 u \quad \to \quad \partial\hat{U}/\partial t = -\alpha\omega^2\hat{U}$$

- Laplacian → multiplication by $-\omega^2$

Spectrograms and Time-Frequency Analysis

Short-Time Fourier Transform (STFT)

$$STFT(t,\omega) = \int f(\tau)w(\tau-t)e^{-i\omega\tau}\, d\tau$$

- **$w(\tau-t)$**: Window function centered at t
- **Meaning**: Fourier transform of windowed signal
- **Result**: Time-frequency representation
- **Trade-off**: Time resolution vs frequency resolution

Wavelet Transform

```
W(a,b) = ∫ f(t)ψ*((t-b)/a) dt
```

- ψ: Mother wavelet (localized in time and frequency)
- **a**: Scale (frequency)
- **b**: Translation (time)
- **Advantage**: Better time-frequency localization

Uncertainty Principle

Time-Frequency Uncertainty

```
Δt·Δω ≥ 1/2
```

- Cannot have perfect localization in both time and frequency
- Narrow in time → wide in frequency
- Narrow in frequency → wide in time
- Fundamental limit (not just measurement)

Example: - Short pulse (narrow Δt) → broad spectrum (large Δω) - Pure tone (narrow Δω) → must last long time (large Δt)

Practice Problems

1. **Cosine transform**: Compute $\mathcal{F}[\cos(\omega_0 t)]$ using Euler's formula.
2. **Gaussian**: Verify that Gaussian is its own Fourier transform (up to scaling).
3. **Convolution**: If f(t) = e^{-t^2} and g(t) = e^{-t^2}, find $\mathcal{F}$[f * g].
4. **Sampling**: What is the Nyquist frequency for sampling rate 1000 Hz?
5. **DFT**: Compute 4-point DFT of [1, 0, 0, 0] by hand.

Key Takeaways

- Fourier transform decomposes signals into frequency components
- Complex exponentials e^{iωt} are basis functions
- F(ω) is complex: magnitude = amplitude, phase = time shift
- Convolution in time = multiplication in frequency
- FFT makes computation practical (O(N log N))
- Fundamental tool in signal processing, quantum mechanics, PDEs
- Time-frequency uncertainty: cannot localize perfectly in both

Historical Note

Jean-Baptiste Joseph Fourier (1822): "Théorie analytique de la chaleur" - Studied heat equation - Claimed any function can be represented as sum of sines and cosines - Controversial at the time - **Dirichlet** (1829): Proved convergence conditions - **Cooley & Tukey** (1965): FFT algorithm - Foundation of modern signal processing

Course Complete!

Congratulations! You've completed the Foundations of Mathematical Intuition course. You can now: - Understand fundamental operations and their meanings - Read equations across multiple mathematical domains - Navigate different number systems and spaces - Apply intuition to VQE and quantum optimization - Decode complex equations from physics, ML, and quantum computing

You've achieved true mathematical literacy. Keep practicing, and remember: every equation tells a story.

Mathematical Intuition · Module 6 — Capstone · Node 6.6
Concepts: inner-product, spectral-analysis, orthogonality, hilbert-space

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