Complex Vector Spaces

Quantum Foundations · Module 1 — Hilbert Spaces

Node 1.1

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Hilbert Spaces **Node:** 1.1

This is the foundation node of the entire course. Every subsequent structure — inner products, operators, observables, states, measurements, entanglement — is built on top of complex vector spaces. Real vector spaces are insufficient for quantum mechanics; the complex field is not optional.

Formal Definition

A **complex vector space** is a set V equipped with two operations:

1. Vector addition: $+ : V \times V \to V$
2. Scalar multiplication: $\cdot : \mathbb{C} \times V \to V$

satisfying, for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$:

- **Associativity:** $(u + v) + w = u + (v + w)$
- **Commutativity:** $u + v = v + u$
- **Zero element:** there exists $0 \in V$ such that $v + 0 = v$
- **Additive inverse:** for each $v \in V$ there exists $-v \in V$ with $v + (-v) = 0$
- **Distributivity (vector):** $\alpha(u + v) = \alpha u + \alpha v$
- **Distributivity (scalar):** $(\alpha + \beta)v = \alpha v + \beta v$
- **Compatibility:** $\alpha(\beta v) = (\alpha\beta)v$
- **Identity:** $1 \cdot v = v$

A **basis** of V is a linearly independent set $\{e_i\} \subset V$ whose linear span equals V. The cardinality of any basis is the **dimension** of V, denoted $\dim V$.

Intuition

A complex vector space is a structure in which:

- Things can be added.
- Things can be scaled by complex numbers.
- The result is again a thing of the same kind.

The complex field is essential because quantum mechanical amplitudes interfere — and interference requires both magnitude and phase. A real vector space carries no notion of phase. This is the structural reason quantum mechanics cannot be formulated over $\mathbb{R}$.

Structural Role

Complex vector spaces are the carrier of quantum states. The state postulate (Node 2.1) will identify physical states with rays in a complex vector space, but the linear-algebraic structure is logically prior.

Every structural object in this course — operators, observables, measurements, channels, entropies — is ultimately a function of, or operator on, a complex vector space.

Mathematical Development

Linear combinations

For $v_1, \dots, v_n \in V$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, the **linear combination** is:

$$\sum_{i=1}^{n} \alpha_i v_i \in V.$$

Linear independence

A set $\{v_1, \dots, v_n\}$ is **linearly independent** if

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \implies \alpha_1 = \cdots = \alpha_n = 0.$$

Span and dimension

The **span** of a set $S \subset V$ is the set of all finite linear combinations of its elements. V is **finite-dimensional** if it has a finite basis; otherwise it is **infinite-dimensional**.

The standard example

$\mathbb{C}^n$ — the set of column vectors with n complex entries — is the prototype finite-dimensional complex vector space. The standard basis is $\{e_1, \dots, e_n\}$ with e_i having a 1 in position i and 0 elsewhere.

Worked Example

The qubit space

The simplest non-trivial quantum system uses $V = \mathbb{C}^2$. A general element is

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}.$$

The standard basis is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A general qubit vector is the linear combination

$$v = \alpha|0\rangle + \beta|1\rangle.$$

This already exhibits the central feature of quantum mechanics: a system can exist in a superposition of basis states with arbitrary complex coefficients.

Cross-Links

- **Linear Algebra:** Vector Spaces (Module 1) — same axiom system, real-or-complex field.
- **Statistics:** Probability simplex is a real convex set; quantum state space replaces it with a complex projective space.
- **VQA:** Parameterized circuits act on $\mathbb{C}^{2^n}$; the underlying linearity is what makes gradient-based optimization tractable.
- **Quantum Information:** Qubits, qudits, and quantum registers are all built directly on this object.

Structural Compression

Invariant: Closure under complex linear combination.

Mechanism: Two operations (addition, complex scalar multiplication) plus eight axioms.

Cross-System Echo: The same axioms over $\mathbb{R}$ give classical state spaces. Replacing $\mathbb{R}$ with $\mathbb{C}$ is the minimal structural perturbation that produces quantum behavior.

Exercises

1. Show that $\mathbb{C}^n$ satisfies all eight axioms of a complex vector space.
2. Prove that the zero element of a vector space is unique.
3. Show that the set of polynomials of degree $\leq n$ with complex coefficients forms a complex vector space, and identify its dimension.
4. Show that $\mathbb{C}$ is itself a complex vector space, and a real vector space — and that its dimension differs in the two cases.
5. Argue, structurally, why a real vector space cannot support the phenomenon of interference.

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Concepts: vector-space, dimensionality
Enables: 1.2, 1.3

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