# Inner Products and Norms

Quantum Foundations · Module 1 — Hilbert Spaces

Node 1.2

## Position in Knowledge Graph

**System:** Quantum Foundations **Structural Domain:** Hilbert Spaces **Node:** 1.2

Inner products turn a complex vector space into a geometric object. Length, angle, and orthogonality become well-defined. The Born rule (Node 2.3) is fundamentally a statement about inner products — every probability in quantum mechanics is the squared modulus of one.

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## Formal Definition

A **complex inner product** on a complex vector space $V$ is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

satisfying, for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$:

- **Conjugate symmetry:** $\langle u, v \rangle = \overline{\langle v, u \rangle}$
- **Linearity in the second argument (physics convention):** $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$
- **Positive definiteness:** $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$

A pair $(V, \langle \cdot, \cdot \rangle)$ is called a **complex inner product space**.

The **induced norm** is

$$\|v\| := \sqrt{\langle v, v \rangle}.$$

Two vectors $u, v$ are **orthogonal**, written $u \perp v$, if $\langle u, v \rangle = 0$. A vector is **normalized** (or a **unit vector**) if $\|v\| = 1$.

Note: physics uses the right-linear convention (linear in the second slot, antilinear in the first). Mathematics texts often use the opposite. Throughout this course we use the physics convention exclusively.

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## Intuition

An inner product is the complex generalization of the dot product. It measures "how much one vector lies along another," but with two twists:

- The output is complex, not real. The complex phase of $\langle \phi, \psi \rangle$ encodes interference — the distinguishing feature of quantum mechanics.
- Only the squared modulus $|\langle \phi, \psi \rangle|^2$ is directly physical, because only it is basis-independent and real-valued. This will become the Born rule.

The norm $\|v\|$ is the length of $v$; the inner product $\langle \phi, \psi \rangle$ is the amplitude with which $\phi$ and $\psi$ overlap.

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## Structural Role

Inner products are what convert the formal linear structure of a complex vector space into the probabilistic structure of quantum mechanics.

- Every quantum probability is $|\langle \phi | \psi \rangle|^2$.
- Every expectation value is $\langle \psi | A | \psi \rangle$.
- The adjoint operation (Node 1.5) is defined with respect to the inner product.
- The spectral theorem (Node 1.6) requires orthogonality of eigenspaces, which requires an inner product.

Without inner products, the course halts at linear algebra and never reaches physics.

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## Mathematical Development

### Cauchy–Schwarz inequality

For all $u, v \in V$,

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|,$$

with equality if and only if $u$ and $v$ are linearly dependent.

### Triangle inequality

For all $u, v \in V$,

$$\|u + v\| \leq \|u\| + \|v\|.$$

This is a direct consequence of Cauchy–Schwarz.

### Parallelogram law

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

This identity characterizes norms that come from an inner product: a norm on a complex vector space is induced by an inner product if and only if it satisfies the parallelogram law.

### Polarization identity

The inner product can be recovered from the norm:

$$\langle u, v \rangle = \tfrac{1}{4} \sum_{k=0}^{3} i^k \|u + i^k v\|^2.$$

Inner product and norm carry the same information — the norm just packages it with the phase discarded.

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## Worked Example

### Qubit inner products

Work on $V = \mathbb{C}^2$ with the standard inner product $\langle u, v\rangle = \sum_i \overline{u_i} v_i$.

Computational basis:

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

Directly:

$$\langle 0|0\rangle = 1, \quad \langle 1|1\rangle = 1, \quad \langle 0|1\rangle = 0.$$

Hadamard basis:

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Compute:

$$\langle +|-\rangle = \tfrac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle) = \tfrac{1}{2}(1 - 0 + 0 - 1) = 0.$$

So $\{|+\rangle, |-\rangle\}$ is orthonormal. Now a cross-basis overlap:

$$\langle 0|+\rangle = \tfrac{1}{\sqrt{2}}(\langle 0|0\rangle + \langle 0|1\rangle) = \tfrac{1}{\sqrt{2}}.$$

Its squared modulus is $|\langle 0|+\rangle|^2 = \frac{1}{2}$ — the Born-rule probability of obtaining outcome 0 when measuring a qubit prepared in state $|+\rangle$ in the computational basis.

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## Cross-Links

- **Linear Algebra:** Inner products and orthogonality — same axiom system over $\mathbb{R}$ or $\mathbb{C}$.
- **Statistics:** Inner products on $L^2$ underlie covariance, correlation, and conditional expectation.
- **VQA:** Cost functions have the form $\langle\psi(\theta)|H|\psi(\theta)\rangle$ — a single inner product.
- **Quantum Information:** Fidelity $F(\psi, \phi) = |\langle\psi|\phi\rangle|^2$ is an inner product quantity.

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## Structural Compression

**Invariant:** A conjugate-symmetric, right-linear, positive-definite pairing $V \times V \to \mathbb{C}$.

**Mechanism:** Three axioms (conjugate symmetry, linearity, positivity) induce a norm, a geometry, and a probability.

**Cross-System Echo:** Real inner products give classical Euclidean geometry. The complex generalization retains lengths and angles but adds phase — the structural ingredient that produces interference.

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## Exercises

1. Verify the three inner-product axioms for the standard inner product $\langle u, v\rangle = \sum_i \overline{u_i} v_i$ on $\mathbb{C}^n$.
2. Prove Cauchy–Schwarz by considering $\|u - \alpha v\|^2 \geq 0$ for an optimally chosen $\alpha$.
3. Verify the parallelogram law on $\mathbb{C}^2$ for $u = |0\rangle, v = |+\rangle$.
4. Show that the polarization identity recovers $\langle u, v\rangle$ from the norm on $\mathbb{C}^2$, using explicit vectors.
5. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Show that $\|\psi\| = 1 \iff |\alpha|^2 + |\beta|^2 = 1$, and interpret the result as a normalization condition on a quantum state.
6. Explain, structurally, why only $|\langle\phi|\psi\rangle|^2$ (not $\langle\phi|\psi\rangle$ itself) appears in measurable quantities.

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*Quantum Foundations · Module 1 — Hilbert Spaces · Node 1.2*
**Concepts:** inner-product, orthogonality
**Prerequisites:** 1.1
**Enables:** 1.3, 1.4