Hilbert Space — Completeness

Quantum Foundations · Module 1 — Hilbert Spaces

Node 1.3

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Hilbert Spaces **Node:** 1.3

This node is the structural promotion of "complex inner product space" to "Hilbert space." In finite dimensions the distinction is vacuous; in infinite dimensions it is what makes quantum mechanics analytically well-posed.

Formal Definition

A **Hilbert space** $\mathcal{H}$ is a complex inner product space that is **complete** in the metric induced by its norm: every Cauchy sequence in $\mathcal{H}$ converges to a limit in $\mathcal{H}$.

Explicitly: a sequence $\{v_n\} \subset \mathcal{H}$ is **Cauchy** if for every $\varepsilon > 0$ there exists N such that

$$\|v_n - v_m\| < \varepsilon \quad \text{for all } n, m \geq N.$$

$\mathcal{H}$ is complete if every such sequence has a limit $v \in \mathcal{H}$ with $\|v_n - v\| \to 0$.

A Hilbert space is **separable** if it contains a countable dense subset. All Hilbert spaces arising in ordinary quantum mechanics are separable.

Intuition

Completeness is the assertion that "limits live in the space." A rational-number line has holes ($\sqrt{2}$ is missing); the real line does not. In the same way, an inner product space that is not complete has sequences whose "intended limit" is outside the space — physically unacceptable, because nothing prevents you from constructing such a sequence as a limit of experimentally realizable states.

The upgrade from "complex inner product space" to "Hilbert space" is the analytic closure that lets you take limits of superpositions, integrals of wavefunctions, and spectral decompositions of unbounded operators without leaving the state space.

In finite dimensions, every inner product space is automatically complete, so the distinction is invisible. In infinite dimensions — the setting for position, momentum, and field operators — it is essential.

Structural Role

Hilbert space is the arena in which all of quantum mechanics takes place. The state postulate (Node 2.1) identifies physical states with unit rays in a Hilbert space; the evolution postulate requires unitary operators, whose definition presumes an inner product and completeness; the measurement postulate demands spectral decomposition, which requires completeness to generalize beyond finite dimensions.

Every downstream object — density matrices (5.1), quantum channels (7.3), partial traces (5.2), Lindblad generators (8.2) — is built on Hilbert space.

Mathematical Development

The three canonical examples

Finite-dimensional: $\mathbb{C}^n$ with $\langle u, v\rangle = \sum_i \overline{u_i} v_i$. Automatically complete.

Countably infinite-dimensional: $\ell^2(\mathbb{N})$, the space of square-summable complex sequences:

$$\ell^2 = \left\{(a_1, a_2, \ldots) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\right\}$$

with $\langle a, b\rangle = \sum_n \overline{a_n} b_n$.

Continuous: $L^2(\mathbb{R})$, the space of square-integrable complex-valued functions modulo almost-everywhere equality:

$$L^2(\mathbb{R}) = \left\{\psi : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx < \infty\right\}$$

with $\langle \phi, \psi\rangle = \int_{-\infty}^{\infty} \overline{\phi(x)} \psi(x) \, dx$.

The last two are separable, infinite-dimensional, and complete — the non-trivial cases.

Separability and basis existence

Every separable Hilbert space admits a countable orthonormal basis (Node 1.4). In non-separable Hilbert spaces an orthonormal basis still exists (by Zorn), but it need not be countable. Separability is what keeps quantum mechanics tractable.

Finite vs infinite dimensions

Property	Finite-dim	Infinite-dim
Completeness automatic	yes	no — must be imposed
Every linear operator bounded	yes	no
Spectral theorem in elementary form	yes	replaced by spectral measure
Position/momentum operators	n/a	unbounded, densely defined

The finite-dimensional case is sufficient for qubit and finite-level physics. The infinite-dimensional case is necessary as soon as position, momentum, or field degrees of freedom appear.

Worked Example

The qubit Hilbert space

For a single qubit, $\mathcal{H} = \mathbb{C}^2$. Two complex dimensions. Every sequence of unit vectors $|\psi_n\rangle$ with $\||\psi_n\rangle - |\psi_m\rangle\| \to 0$ has a limit in $\mathbb{C}^2$ — there is no analytic subtlety.

The particle-on-a-line Hilbert space

For a non-relativistic particle on $\mathbb{R}$, $\mathcal{H} = L^2(\mathbb{R})$. Elements are wavefunctions $\psi(x)$ with $\int |\psi|^2\,dx < \infty$. The Gaussian $\psi_\sigma(x) = (2\pi\sigma^2)^{-1/4} e^{-x^2/(4\sigma^2)}$ is in $\mathcal{H}$ for every $\sigma > 0$. A sequence of Gaussians with $\sigma_n \to \sigma_\infty > 0$ has a Gaussian limit, which is again in $\mathcal{H}$. A sequence with $\sigma_n \to 0$ has no limit in L^2 — the pointwise limit is a Dirac delta, which is not square-integrable. The latter is why $\delta(x - x_0)$ is not a physical state; it lives outside the Hilbert space, and position "eigenstates" are a formal device (rigged Hilbert space), not genuine vectors.

Cross-Links

- **Linear Algebra:** Vector spaces with inner products — the finite-dimensional case of this node.
- **Statistics:** L^2 is the Hilbert space of square-integrable random variables; conditional expectation is Hilbert-space projection.
- **VQA:** An n-qubit register lives in $(\mathbb{C}^2)^{\otimes n}$ — a finite-dimensional Hilbert space of dimension 2^n.
- **Math-Intuition:** Completeness in analysis — the same idea specialized to the complex-inner-product setting.

Structural Compression

Invariant: Cauchy-completeness in the inner-product norm.

Mechanism: Topological closure of an inner product space under limits of Cauchy sequences.

Cross-System Echo: Banach completeness in classical analysis generalizes to arbitrary norms; Hilbert completeness is the special case with an inner product structure, enabling orthogonality and spectral theory.

Exercises

1. Prove directly that $\mathbb{C}^n$ is complete, using that $\mathbb{C}$ is complete and that convergence in $\mathbb{C}^n$ reduces to coordinate-wise convergence.
2. Give an example of an incomplete inner product space. (Hint: consider the space of eventually-zero sequences as a subspace of ℓ^2.)
3. Show that ℓ^2 is complete. Show that it is separable by exhibiting a countable dense subset.
4. Show that the Dirac delta $\delta(x)$ is not an element of $L^2(\mathbb{R})$.
5. Let $\mathcal{H}$ be separable and $\{e_n\}$ an orthonormal basis. Show that the map $\mathcal{H} \to \ell^2$, $v \mapsto (\langle e_n, v\rangle)$, is an isometric isomorphism. Conclude that there is essentially only one separable infinite-dimensional Hilbert space.
6. Explain structurally why position eigenstates are "formal" rather than genuine elements of the Hilbert space, and why this does not prevent their use in calculations.

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Concepts: hilbert-space, convergence
Prerequisites: 1.2
Enables: 1.4, 1.5

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