Orthonormal Bases and Dirac Notation

Quantum Foundations · Module 1 — Hilbert Spaces

Node 1.4

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Hilbert Spaces **Node:** 1.4

This node introduces the bra-ket notation that the rest of the course uses without further comment. It also establishes the resolution of identity, the structural identity that makes basis expansions tractable and is the engine of every spectral decomposition and measurement computation.

Formal Definition

Let $\mathcal{H}$ be a separable Hilbert space.

An **orthonormal set** is a collection $\{|e_i\rangle\} \subset \mathcal{H}$ satisfying

$$\langle e_i|e_j\rangle = \delta_{ij}.$$

An orthonormal set is an **orthonormal basis** if it is additionally **complete**: its linear span is dense in $\mathcal{H}$, equivalently,

$$\sum_i |e_i\rangle\langle e_i| = I,$$

where the sum converges in the strong operator topology (in finite dimensions, as an ordinary sum).

Dirac notation

Vectors in $\mathcal{H}$ are written as **kets**: $|\psi\rangle \in \mathcal{H}$.

The **dual space** $\mathcal{H}^*$ consists of continuous linear functionals on $\mathcal{H}$. By the **Riesz representation theorem**, every $f \in \mathcal{H}^*$ has a unique representer $\phi \in \mathcal{H}$ such that $f(\psi) = \langle \phi, \psi\rangle$. The dual element associated with $|\phi\rangle$ is written as a **bra**: $\langle\phi| \in \mathcal{H}^*$.

The pairing is the inner product:

$$\langle\phi|\psi\rangle \equiv \langle\phi, \psi\rangle.$$

A bra next to a ket is a scalar; a ket next to a bra is an operator:

$$|\phi\rangle\langle\psi| : |\chi\rangle \mapsto |\phi\rangle\, \langle\psi|\chi\rangle.$$

Intuition

Dirac notation is a disciplined typographic device that automatically tracks “row vs column” and “vector vs linear-functional”:

- Ket $|\psi\rangle$: column vector, element of $\mathcal{H}$.
- Bra $\langle\psi|$: row vector, element of $\mathcal{H}^*$.
- Bra-ket $\langle\phi|\psi\rangle$: scalar, the inner product.
- Ket-bra $|\phi\rangle\langle\psi|$: rank-one operator.

The notation enforces the typing: any expression that can be written down in Dirac notation is syntactically valid, and its “type” (scalar, vector, operator) is visible from the bracket pattern.

An orthonormal basis is a coordinate system on $\mathcal{H}$. The completeness relation $\sum_i |e_i\rangle\langle e_i| = I$ is what lets you insert a basis expansion anywhere in a computation, the way you insert “1” in an arithmetic identity.

Structural Role

The resolution of identity

$$I = \sum_i |e_i\rangle\langle e_i|$$

is the engine behind every basis expansion, every measurement probability, and every spectral decomposition. Given any state $|\psi\rangle$,

$$|\psi\rangle = I|\psi\rangle = \sum_i |e_i\rangle\langle e_i|\psi\rangle,$$

so the coefficients in the basis expansion are exactly the inner products $\langle e_i|\psi\rangle$. These coefficients, squared, are Born-rule probabilities (Node 2.3). The resolution of identity therefore structurally contains all of measurement statistics as a special case.

Mathematical Development

Basis expansion

For any $|\psi\rangle \in \mathcal{H}$:

$$|\psi\rangle = \sum_i c_i|e_i\rangle, \qquad c_i = \langle e_i|\psi\rangle.$$

Parseval’s identity

$$\|\psi\|^2 = \sum_i |c_i|^2 = \sum_i |\langle e_i|\psi\rangle|^2.$$

This is the Hilbert-space Pythagorean theorem: the squared length of a vector is the sum of the squared moduli of its basis coefficients.

Inner product in coordinates

$$\langle\phi|\psi\rangle = \sum_i \overline{\langle e_i|\phi\rangle}\,\langle e_i|\psi\rangle = \sum_i \langle\phi|e_i\rangle\langle e_i|\psi\rangle.$$

Change of basis

Let $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ be two orthonormal bases. The change-of-basis operator is

$$U = \sum_j |f_j\rangle\langle e_j|,$$

which satisfies $U^\dagger U = UU^\dagger = I$ — a unitary operator (Node 1.5). A state's coefficients transform as

$$\langle f_j|\psi\rangle = \sum_i \langle f_j|e_i\rangle\langle e_i|\psi\rangle,$$

i.e., multiplication by the matrix $U_{ji} = \langle f_j|e_i\rangle$.

Gram–Schmidt

Any countable linearly independent set $\{v_n\}$ in $\mathcal{H}$ can be converted into an orthonormal set by the Gram–Schmidt procedure:

$$|u_n\rangle = |v_n\rangle - \sum_{k<n} |e_k\rangle\langle e_k|v_n\rangle, \qquad |e_n\rangle = \frac{|u_n\rangle}{\|u_n\|}.$$

This is the constructive proof that every separable Hilbert space has an orthonormal basis.

Worked Example

Computational and Hadamard bases on $\mathbb{C}^2$

Work on the qubit Hilbert space $\mathbb{C}^2$. The **computational basis** is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Verify the completeness relation:

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

The **Hadamard basis** is

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Change of basis:

$$|0\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \qquad |1\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

The change-of-basis operator is the **Hadamard matrix**

$$H = |+\rangle\langle 0| + |-\rangle\langle 1| = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which satisfies $H^\dagger H = I$ and $H^2 = I$.

Expanding a state in two bases

Consider $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$. In the computational basis the coefficients are $(1/\sqrt{3}, \sqrt{2/3})$, with squared moduli $(1/3, 2/3)$ — the Born probabilities for outcomes 0 and 1 in a computational measurement.

Hadamard coefficients:

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}}\right), \quad \langle -|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}} - \sqrt{\frac{2}{3}}\right).$$

Parseval: $|\langle +|\psi\rangle|^2 + |\langle -|\psi\rangle|^2 = 1$. The probabilities of the two outcomes depend on the measurement basis, but their sum is invariant — this is the structural reason probability is conserved under change of basis.

Cross-Links

- **Linear Algebra:** Orthonormal bases and Gram–Schmidt.
- **Quantum Information:** Computational vs Hadamard bases are the standard measurement bases for protocols (e.g., BB84).
- **VQA:** Measurement expansion in the computational basis — every cost-function estimate is a sum over basis-projector expectation values.
- **Statistics:** Fourier basis on $L^2(S^1)$ as an example of an infinite-dimensional orthonormal basis; characteristic functions.

Structural Compression

Invariant: The resolution of identity $\sum_i |e_i\rangle\langle e_i| = I$.

Mechanism: Orthonormality plus completeness — two-part axiom.

Cross-System Echo: Fourier expansion in L^2; character sums in harmonic analysis; principal component decomposition in statistics. All are instances of "expand in an orthonormal basis that diagonalizes the structure of interest."

Exercises

1. Verify that $\{|+\rangle, |-\rangle\}$ is an orthonormal basis of $\mathbb{C}^2$: check $\langle +|+\rangle = \langle -|-\rangle = 1$, $\langle +|-\rangle = 0$, and the completeness relation $|+\rangle\langle +| + |-\rangle\langle -| = I$.
2. Prove Parseval's identity from the resolution of identity.
3. Apply Gram–Schmidt to the vectors $v_1 = (1,0)^T$ and $v_2 = (1,1)^T$ in $\mathbb{C}^2$. Check that the result is orthonormal.
4. Show that the Hadamard matrix H satisfies $H = H^\dagger$ and $H^2 = I$. Interpret: H is both a change of basis and its own inverse.
5. For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, compute the coefficients in the Hadamard basis. Verify that the squared moduli sum to $|\alpha|^2 + |\beta|^2$.
6. Explain, structurally, why the completeness relation is the single identity that generates both basis expansions and measurement probabilities.

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Concepts: orthogonality, hilbert-space, dimensionality

Prerequisites: 1.3
Enables: 1.5, 2.1

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