# Linear Operators and Adjoints

Quantum Foundations · Module 1 — Hilbert Spaces

Node 1.5

## Position in Knowledge Graph

**System:** Quantum Foundations **Structural Domain:** Hilbert Spaces **Node:** 1.5

The objects acting on quantum states are linear operators. The adjoint operation, defined through the inner product, distinguishes the special operator classes — Hermitian (observables), unitary (evolution), and projection (measurement) — that the postulates of quantum mechanics require.

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## Formal Definition

Let $\mathcal{H}$ be a Hilbert space.

A **linear operator** on $\mathcal{H}$ is a map $A : \mathcal{H} \to \mathcal{H}$ satisfying

$$A(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha A|\psi\rangle + \beta A|\phi\rangle$$

for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.

The operator $A$ is **bounded** if there exists $C \geq 0$ with $\|A|\psi\rangle\| \leq C\|\psi\|$ for all $|\psi\rangle$. The smallest such $C$ is the **operator norm** $\|A\|$. In finite dimensions every linear operator is bounded; in infinite dimensions, operators like position and momentum are unbounded and only densely defined.

The **adjoint** $A^\dagger$ of a bounded linear operator $A$ is the unique bounded linear operator satisfying

$$\langle\phi|A\psi\rangle = \langle A^\dagger\phi|\psi\rangle \quad \text{for all } |\phi\rangle, |\psi\rangle \in \mathcal{H}.$$

Existence and uniqueness follow from the Riesz representation theorem applied to the functional $|\psi\rangle \mapsto \langle\phi|A\psi\rangle$.

### Operator classes

- **Hermitian (self-adjoint):** $A = A^\dagger$. Corresponds to observables.
- **Unitary:** $U^\dagger U = UU^\dagger = I$. Corresponds to physical evolution and change of basis.
- **Projection (orthogonal):** $P^2 = P$ and $P^\dagger = P$. Corresponds to yes/no measurement questions.
- **Normal:** $[A, A^\dagger] = 0$. The largest class admitting the spectral theorem.

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## Intuition

Operators are "things that do something to states." They are the verbs of quantum mechanics; states are the nouns.

The adjoint is the natural generalization of the matrix conjugate-transpose. In finite dimensions, if $A$ is represented in an orthonormal basis by the matrix $A_{ij} = \langle e_i|A|e_j\rangle$, then $A^\dagger_{ij} = \overline{A_{ji}}$. Conjugate, then transpose.

The three distinguished classes correspond to the three things one does with a quantum state:

- **Hermitian $\leftrightarrow$ real eigenvalues $\leftrightarrow$ measurable quantities.** Eigenvalues are the possible outcomes of a measurement, and measurement outcomes are real numbers.
- **Unitary $\leftrightarrow$ norm-preserving $\leftrightarrow$ probability-conserving evolution.** Unitaries are invertible and preserve inner products, which is what makes Born-rule probabilities a conserved total.
- **Projection $\leftrightarrow$ idempotent yes/no $\leftrightarrow$ measurement outcomes as subspaces.** A projection partitions the Hilbert space into "inside the subspace" and "orthogonal to it" — the two possibilities of a yes/no test.

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## Structural Role

The three operator classes each carry a postulate of quantum mechanics:

- Hermitian operators represent observables (Postulate 2, Node 2.2).
- Unitary operators represent closed-system time evolution (Postulate 4, Node 2.4).
- Projections (and their generalizations, POVMs) represent measurements (Postulates 3, 3.1, 3.2).

The adjoint is the single algebraic device that organizes these distinctions: all three classes are defined by adjoint equations. Without the inner product (Node 1.2), none of these definitions exist.

Downstream, the adjoint is also the structural origin of quantum channels (Node 7.3), because Kraus operators arise as pairs $\{K_i\}$ with $\sum_i K_i^\dagger K_i = I$ — an adjoint condition that guarantees probability conservation.

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## Mathematical Development

### Adjoint algebra

For bounded operators $A, B$ and scalar $\alpha$:

$$(A+B)^\dagger = A^\dagger + B^\dagger, \quad (\alpha A)^\dagger = \overline{\alpha} A^\dagger, \quad (AB)^\dagger = B^\dagger A^\dagger, \quad (A^\dagger)^\dagger = A.$$

Note the order reversal in $(AB)^\dagger$ and the complex conjugation of scalars.

### Trace

For a trace-class operator, the **trace** is

$$\mathrm{Tr}(A) = \sum_i \langle e_i|A|e_i\rangle,$$

independent of the orthonormal basis $\{|e_i\rangle\}$. Key properties:

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \quad \mathrm{Tr}(A^\dagger) = \overline{\mathrm{Tr}(A)}.$$

The trace will reappear in Node 5.1 as the normalization condition for density matrices.

### Commutators

The **commutator** $[A, B] := AB - BA$ measures non-commutativity. For Hermitian $A, B$:

$$[A, B]^\dagger = -[A, B],$$

so commutators of Hermitians are anti-Hermitian, and $i[A, B]$ is Hermitian. Commutators generate the uncertainty relations of Node 3.5.

### Characterizations of the three classes

- **Hermitian** $\iff$ $\langle\psi|A|\psi\rangle \in \mathbb{R}$ for all $|\psi\rangle$. The expectation value of a Hermitian operator is real, which is why observables must be Hermitian.
- **Unitary** $\iff$ $\|U|\psi\rangle\| = \|\psi\|$ for all $|\psi\rangle$. Norm preservation is equivalent to the adjoint condition $U^\dagger U = I$.
- **Projection** $\iff$ $P$ is the orthogonal projection onto some closed subspace $\mathcal{M} \subseteq \mathcal{H}$.

### Rank-one operators

The ket-bra $|\phi\rangle\langle\psi|$ is a rank-one operator with

$$(|\phi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\phi|.$$

In particular, $|\psi\rangle\langle\psi|$ is a Hermitian rank-one projection onto the one-dimensional subspace spanned by $|\psi\rangle$ (provided $\|\psi\| = 1$).

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## Worked Example

### The Pauli matrices on $\mathbb{C}^2$

The Pauli operators are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**Hermiticity.** Compute $X^\dagger$ by conjugate-transposing: $X^\dagger = X$. Similarly $Y^\dagger = Y$ (note: the off-diagonal entries are $-i$ and $i$, which swap and conjugate back to themselves) and $Z^\dagger = Z$. All three are Hermitian.

**Unitarity.** Each Pauli matrix satisfies $X^2 = Y^2 = Z^2 = I$, so $X^\dagger X = X^2 = I$, and similarly for $Y$ and $Z$. All three are unitary. (A Hermitian operator that squares to the identity is automatically unitary — an uncommon but structurally important coincidence.)

**Algebra.** Direct computation:

$$[X, Y] = XY - YX = 2iZ, \quad [Y, Z] = 2iX, \quad [Z, X] = 2iY.$$

These are the defining relations of $\mathfrak{su}(2)$, the Lie algebra of rotations — the algebraic source of spin-$\frac{1}{2}$ behavior.

**Spectral data.** $Z$ has eigenvalues $\pm 1$ with eigenvectors $|0\rangle, |1\rangle$. $X$ has eigenvalues $\pm 1$ with eigenvectors $|\pm\rangle$. $Y$ has eigenvalues $\pm 1$ with eigenvectors $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$.

**A projection example**

The operator $P_+ = |0\rangle\langle 0|$ satisfies

$$P_+^2 = |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0| = P_+, \quad P_+^\dagger = (|0\rangle\langle 0|)^\dagger = |0\rangle\langle 0| = P_+.$$

It projects onto the one-dimensional subspace $\mathrm{span}\{|0\rangle\}$. Its complement $P_- = I - P_+ = |1\rangle\langle 1|$ is also a projection, and $P_+ + P_- = I$. Together they form the computational-basis measurement (Node 3.1).

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## Cross-Links

- **Linear Algebra:** Linear transformations and adjoints (module on inner product spaces). Matrix transpose is the real special case.
- **Statistics:** Self-adjoint operators on $L^2$ generalize the notion of random variable; covariance operators are Hermitian positive-semidefinite.
- **VQA:** Hamiltonians are Hermitian observables; variational cost functions are inner products $\langle\psi(\theta)|H|\psi(\theta)\rangle$.
- **Quantum Information:** Kraus operators of quantum channels are constrained by $\sum_i K_i^\dagger K_i = I$ — an adjoint relation.

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## Structural Compression

**Invariant:** Linearity, plus the adjoint equation $\langle\phi|A\psi\rangle = \langle A^\dagger\phi|\psi\rangle$.

**Mechanism:** One linearity axiom and one duality relation (via Riesz) carve out the three operator classes needed by the quantum postulates.

**Cross-System Echo:** Matrix transpose is the real special case of the adjoint; orthogonal matrices are the real analogue of unitaries; symmetric matrices are the real analogue of Hermitians. Quantum mechanics is the complex-linear upgrade of this classical structure.

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## Exercises

1. Show that the adjoint is unique: if $B_1$ and $B_2$ both satisfy $\langle\phi|A\psi\rangle = \langle B_k\phi|\psi\rangle$ for all $|\phi\rangle, |\psi\rangle$, then $B_1 = B_2$.
2. Prove that $(AB)^\dagger = B^\dagger A^\dagger$ directly from the defining relation of the adjoint.
3. Prove that the expectation value of a Hermitian operator is real: $\langle\psi|A|\psi\rangle \in \mathbb{R}$ when $A = A^\dagger$.
4. Prove that a linear operator is unitary if and only if it preserves the norm of every vector.
5. Verify the Pauli commutation relations $[X, Y] = 2iZ$, $[Y, Z] = 2iX$, $[Z, X] = 2iY$ by direct matrix calculation.
6. Show that the ket-bra $|\psi\rangle\langle\psi|$ (with $\|\psi\| = 1$) is a projection. Compute its trace and its action on an arbitrary vector.
7. Let $A$ be Hermitian and $U$ unitary. Show that $UAU^\dagger$ is Hermitian and has the same eigenvalues as $A$. Interpret this as "change of basis preserves observables."

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*Quantum Foundations · Module 1 — Hilbert Spaces · Node 1.5*
**Concepts:** linear-operators, matrix-representation, hilbert-space
**Prerequisites:** 1.4
**Enables:** 1.6, 2.2

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