# Spectral Decomposition

Quantum Foundations · Module 1 — Hilbert Spaces

Node 1.6

## Position in Knowledge Graph

**System:** Quantum Foundations **Structural Domain:** Hilbert Spaces **Node:** 1.6

The spectral theorem is the bridge from linear algebra into quantum measurement. Every observable is decomposed into a sum over eigenvalues weighted by orthogonal projections — and this decomposition is exactly what the Born rule (Node 2.3) and projective measurement (Node 3.1) consume. This is the closing node of Module 1; the next module will start converting this machinery into physics.

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## Formal Definition

### Finite-dimensional spectral theorem

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space and $A$ a Hermitian operator on $\mathcal{H}$. Then $A$ admits a **spectral decomposition**

$$A = \sum_i \lambda_i P_i,$$

where:

- $\lambda_i \in \mathbb{R}$ are the distinct eigenvalues of $A$,
- $P_i$ is the orthogonal projection onto the eigenspace $\mathcal{H}_i = \ker(A - \lambda_i I)$,
- the projections are mutually orthogonal: $P_i P_j = \delta_{ij} P_i$,
- the projections resolve the identity: $\sum_i P_i = I$.

The eigenspaces give an orthogonal decomposition $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. The theorem holds more generally for **normal** operators (those satisfying $[A, A^\dagger] = 0$), with complex eigenvalues; Hermiticity is the special case that guarantees real eigenvalues.

### Spectral theorem (general form)

For a bounded normal operator $A$ on a (possibly infinite-dimensional) Hilbert space, the spectral sum is replaced by a **spectral measure**: a projection-valued measure $E$ on the complex plane (supported on the spectrum of $A$) such that

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda).$$

The discrete-sum version is recovered when the spectrum is a countable set of isolated eigenvalues.

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## Intuition

Every observable is a list of real outcomes plus a partition of Hilbert space into "what would collapse to this outcome" subspaces.

Think of the Hermitian operator $A$ as a compound object with two pieces of information:

1. A finite list of real numbers $\lambda_1, \lambda_2, \ldots$ — the possible measurement outcomes.
2. A matching list of orthogonal subspaces — one per outcome — that jointly tile the whole Hilbert space.

A state "lying inside" eigenspace $i$ deterministically yields outcome $\lambda_i$. A state that is a superposition across eigenspaces yields outcome $i$ with probability equal to the squared length of its projection onto that eigenspace. That is the Born rule, prefigured by the spectral theorem.

The spectral theorem is why quantum mechanics is diagonal-in-the-right-basis: for any observable, there is a basis (its eigenbasis) in which measurement is trivial to compute.

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## Structural Role

The spectral theorem is the structural hinge of the course. Everything before it is abstract linear algebra; everything after it is physics.

- It supplies the projections that appear in projective measurement (Node 3.1).
- It guarantees that every observable has real outcomes (Node 2.2).
- It gives the Born rule an algebraic shape: $\Pr(\lambda_i) = \mathrm{Tr}(P_i \rho)$ (Node 2.3, Node 5.1).
- It diagonalizes density matrices into mixtures of pure states (Node 5.1).
- It diagonalizes Hamiltonians into time-evolution-generating eigenmodes (Node 2.4).
- In infinite dimensions, it is what gives position, momentum, and energy their spectra and their associated projective measures.

No spectral theorem, no measurement theory.

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## Mathematical Development

### Proof sketch (finite-dimensional)

Let $A$ be Hermitian on $\mathcal{H}$ with $\dim \mathcal{H} = n < \infty$.

**Step 1 — existence of a real eigenvalue.** The characteristic polynomial $\det(A - \lambda I)$ has at least one root in $\mathbb{C}$. For any eigenvector $|v\rangle$ with $A|v\rangle = \lambda|v\rangle$ and $\|v\| = 1$:

$$\lambda = \langle v|A|v\rangle = \overline{\langle v|A|v\rangle} = \overline{\lambda},$$

using $A = A^\dagger$. So every eigenvalue is real.

**Step 2 — eigenspaces are orthogonal.** If $A|v\rangle = \lambda|v\rangle$ and $A|w\rangle = \mu|w\rangle$ with $\lambda \neq \mu$, then

$$\lambda\langle w|v\rangle = \langle w|Av\rangle = \langle Aw|v\rangle = \mu\langle w|v\rangle,$$

forcing $\langle w|v\rangle = 0$.

**Step 3 — induction on dimension.** Let $\mathcal{H}_\lambda = \ker(A - \lambda I)$ be the eigenspace for one eigenvalue $\lambda$. Its orthogonal complement $\mathcal{H}_\lambda^\perp$ is invariant under $A$: for $|w\rangle \in \mathcal{H}_\lambda^\perp$ and $|v\rangle \in \mathcal{H}_\lambda$,

$$\langle v|Aw\rangle = \langle Av|w\rangle = \lambda\langle v|w\rangle = 0,$$

so $A|w\rangle \in \mathcal{H}_\lambda^\perp$. Restrict $A$ to $\mathcal{H}_\lambda^\perp$, which is strictly lower-dimensional, and apply the induction hypothesis.

**Step 4 — assemble.** Iterating gives an orthogonal decomposition $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. Set $P_i$ to be the orthogonal projection onto $\mathcal{H}_i$. Then $P_i P_j = \delta_{ij} P_i$, $\sum_i P_i = I$, and $A = \sum_i \lambda_i P_i$ because both sides agree on a basis.

### Functional calculus

Once $A = \sum_i \lambda_i P_i$, one defines functions of operators by applying the function to eigenvalues:

$$f(A) := \sum_i f(\lambda_i) P_i.$$

This gives exponentials $e^{-iHt/\hbar} = \sum_i e^{-i\lambda_i t/\hbar} P_i$ (the time-evolution operator of Node 2.4), powers, square roots, and logarithms — all constructed directly from the spectral data.

### Unitary diagonalization

Equivalently: every Hermitian $A$ can be written as $A = UDU^\dagger$ where $D$ is diagonal with the eigenvalues of $A$ on the diagonal and $U$ is the unitary whose columns are the eigenvectors of $A$. The two forms are equivalent: $P_i = U\Pi_i U^\dagger$ where $\Pi_i$ is the diagonal projection onto coordinate $i$ in the standard basis.

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## Worked Example

### Pauli $Z$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Eigenvalues: $+1$ with eigenvector $|0\rangle$, and $-1$ with eigenvector $|1\rangle$. Projections: $P_+ = |0\rangle\langle 0|$, $P_- = |1\rangle\langle 1|$. Spectral decomposition:

$$Z = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|.$$

Verify the resolution of identity: $P_+ + P_- = |0\rangle\langle 0| + |1\rangle\langle 1| = I$. Verify orthogonality: $P_+ P_- = |0\rangle\langle 0|1\rangle\langle 1| = 0$.

### Pauli $X$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Eigenvalues: $+1$ with eigenvector $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $-1$ with eigenvector $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Spectral decomposition:

$$X = (+1)|+\rangle\langle +| + (-1)|-\rangle\langle -|.$$

In matrix form: $|+\rangle\langle +| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Their difference reproduces $X$; their sum is $I$.

**Functional calculus on $Z$**

Compute $e^{-iZt}$:

$$e^{-iZt} = e^{-it}|0\rangle\langle 0| + e^{+it}|1\rangle\langle 1| = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{+it} \end{pmatrix}.$$

This is the time-evolution operator for a qubit Hamiltonian $H = Z$ (taking $\hbar = 1$). The exponential is trivial in the eigenbasis — the spectral theorem is what lets you compute it.

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## Cross-Links

- **Linear Algebra:** The spectral theorem is the signature theorem of the linear algebra course. This node is its complex-Hermitian specialization.
- **Statistics:** Principal component analysis is the spectral decomposition of a covariance matrix. The same machinery; different interpretation.
- **VQA:** Variational eigensolvers target the smallest eigenvalue of a spectral decomposition via parameterized circuits; the ground state is the $\lambda_{\min}$ eigenvector.
- **Quantum Information:** Von Neumann entropy (Node 7.4) is computed by spectrally decomposing a density matrix.
- **Quantum Thermodynamics:** The Gibbs state $e^{-\beta H}/Z$ is defined by the functional calculus applied to the Hamiltonian.

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## Structural Compression

**Invariant:** Every Hermitian operator equals an eigenvalue-weighted sum of mutually orthogonal projections that resolve the identity.

**Mechanism:** Orthogonality of eigenspaces (from the adjoint relation) plus induction on dimension (finite case), or projection-valued measures (infinite case).

**Cross-System Echo:** Diagonalization of symmetric matrices in real linear algebra; Fourier analysis as the spectral decomposition of the translation operator; principal components as the spectral decomposition of a covariance operator. All are instances of "resolve an operator into outcomes plus a basis adapted to those outcomes."

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## Exercises

1. Compute the spectral decomposition of Pauli $Y$. Write out the projections explicitly as $2 \times 2$ matrices.
2. Let $A$ be Hermitian with spectral decomposition $A = \sum_i \lambda_i P_i$. Prove that $A^2 = \sum_i \lambda_i^2 P_i$ and, more generally, $p(A) = \sum_i p(\lambda_i) P_i$ for any polynomial $p$.
3. Show that $\mathrm{Tr}(A) = \sum_i \lambda_i \dim \mathcal{H}_i$ where $\mathcal{H}_i$ is the eigenspace for $\lambda_i$.
4. Let $A$ be Hermitian on $\mathbb{C}^n$ with spectral decomposition $A = \sum_i \lambda_i P_i$. Show that $\langle\psi|A|\psi\rangle = \sum_i \lambda_i \|P_i|\psi\rangle\|^2$ and interpret this as a weighted average of eigenvalues with weights given by Born-rule probabilities.
5. Construct an operator on $\mathbb{C}^3$ with eigenvalues $(0, 1, 1)$ and a non-trivial two-dimensional eigenspace for the eigenvalue 1. Exhibit its spectral decomposition with two (not three) projections.
6. Show that a projection $P$ is itself a Hermitian operator whose spectral decomposition has only the eigenvalues 0 and 1. What are the two eigenspaces?
7. Explain, structurally, why the spectral theorem is the bridge between "linear algebra" and "measurement theory" in this course.

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*Quantum Foundations · Module 1 — Hilbert Spaces · Node 1.6*
**Concepts:** eigenvalue-decomposition, spectral-analysis, linear-operators
**Prerequisites:** 1.5
**Enables:** 2.2, 2.3, 3.1

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