Bell's Theorem and Nonlocality

Quantum Foundations · Module 10 — Foundations And Open Problems

Node 10.1

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Foundations and Open Problems **Node:** 10.1

This is the foundation node of the entire foundations module. Bell's theorem is the structural watershed of 20th-century physics: it converts a metaphysical question — "is the world locally classical?" — into an experimentally falsifiable statement, and the experiment has been performed. The world is not locally classical.

Every subsequent foundation node in this module sits in the conceptual space Bell's theorem opened.

Formal Statement

Bell's theorem (1964). Let A_0, A_1 be Alice's measurement choices and B_0, B_1 be Bob's, each producing outcomes in $\{-1, +1\}$. Suppose the joint probabilities of outcomes admit a **local hidden-variable** description:

$$p(a, b \mid x, y) = \int p(\lambda)\, p_A(a \mid x, \lambda)\, p_B(b \mid y, \lambda)\, d\lambda,$$

where λ is a hidden variable shared by both parties, sampled from some distribution $p(\lambda)$, and the marginals depend only on each party's local measurement choice.

Then the **CHSH inequality** holds:

$$|E(A_0B_0) + E(A_0B_1) + E(A_1B_0) - E(A_1B_1)| \leq 2.$$

Quantum mechanics violates this bound. For the Bell state $|\Phi^+\rangle$ and an appropriate choice of measurement axes, quantum mechanics predicts

$$|E(A_0B_0) + E(A_0B_1) + E(A_1B_0) - E(A_1B_1)| = 2\sqrt{2}$$

(the **Tsirelson bound**), which exceeds the local hidden-variable bound of 2.

Experimental verification (loophole-free, 2015–present): the prediction is correct. No local hidden-variable theory can reproduce the observed correlations.

What This Rules Out

Bell's theorem rules out a specific structural assumption about the world:

> **Local realism:** physical systems possess definite values for all measurable quantities at all times (realism), and these values cannot be influenced by spacelike-separated events (locality).

The conjunction of these two assumptions is empirically false. At least one must go.

What survives:

- Quantum mechanics (which is non-local in the sense of correlations, but not in the sense of usable signalling).
- Bohmian mechanics (which is explicitly non-local but realist).
- Many-worlds (which is local at the level of the wavefunction but non-realist about single outcomes).
- QBism (which gives up "outcomes are properties of the world" entirely).

What does not survive:

- Any theory in which measurement reveals pre-existing local properties of the system being measured.

Intuition

Imagine two friends sent to opposite ends of the universe with sealed envelopes. Locally, each one can find correlations between what is in their envelope and various contextual variables. The set of correlations that can arise from "they each opened a pre-prepared envelope" is mathematically constrained — and the constraint takes the form of the CHSH inequality.

Quantum mechanics produces correlations that exceed this constraint. Therefore: there is no pre-prepared envelope. The act of measurement is not the act of reading out a value that was already there.

The crucial structural fact is that this is an empirical statement, not a philosophical one. The CHSH inequality is a theorem about correlations in *any* local hidden-variable model; the violation is observed in actual experiments. This is the closest physics has ever come to a formal experimental test of metaphysics.

Structural Role

Bell's theorem is the central organizing fact of this module:

- **Kochen–Specker theorem (10.2)** strengthens it: even non-local hidden variables face structural constraints (contextuality).
- **PBR theorem (10.3)** strengthens it differently: ψ-epistemic models (where the wavefunction represents knowledge about a deeper reality) face structural constraints.
- **Quantum-to-classical transition (10.4)** is constrained by Bell — any reduction must preserve the violation.
- **Measurement problem (10.5)** must be solved consistently with Bell's empirical content.

It is also the empirical foundation of:

- **Device-independent QKD:** if you observe a Bell violation, you know the channel is secure regardless of the hardware details.
- **Quantum random number generators:** Bell-violating outcomes are provably indeterministic.
- **Self-testing of quantum devices:** the observed correlations alone certify the underlying state and measurements.

Mathematical Development

Derivation of the CHSH classical bound

In any local hidden-variable model, the outcomes $a, b \in \{\pm 1\}$ factor through λ. For each fixed λ, the quantity

$$A_0(\lambda)(B_0(\lambda) + B_1(\lambda)) + A_1(\lambda)(B_0(\lambda) - B_1(\lambda))$$

equals ± 2 (one of the two parenthesized expressions is zero, the other is ± 2). Averaging over λ preserves the bound, so

$$|E(A_0B_0) + E(A_0B_1) + E(A_1B_0) - E(A_1B_1)| \leq 2.$$

This is the structural argument: the bound comes from the fact that classical outcomes are ± 1 functions of λ, and a ± 1 function has bounded variation.

Quantum violation

Take $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, Alice's measurements $A_0 = Z, A_1 = X$, Bob's measurements $B_0 = (Z + X)/\sqrt{2}, B_1 = (Z - X)/\sqrt{2}$. Compute:

$$E(A_0B_0) = \langle \Phi^+ | Z \otimes \frac{Z + X}{\sqrt{2}} | \Phi^+ \rangle = \frac{1}{\sqrt{2}}.$$

By symmetry, $E(A_0B_1) = E(A_1B_0) = 1/\sqrt{2}$ and $E(A_1B_1) = -1/\sqrt{2}$. The CHSH expression is then

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\sqrt{2} \approx 2.828.$$

This is the maximum quantum value (Tsirelson 1980); no choice of state or measurement can exceed it.

Worked Example

Numerical Bell test from $|\Phi^+\rangle$

Consider the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and the four Tsirelson-optimal measurement settings:

- Alice: $A_0 = Z$, $A_1 = X$.
- Bob: $B_0 = (Z + X)/\sqrt{2}$, $B_1 = (Z - X)/\sqrt{2}$.

Step 1: Compute exact expectations. For each pair (A_x, B_y), compute $E(A_xB_y) = \langle \Phi^+ | A_x \otimes B_y | \Phi^+ \rangle$ using the identity $\langle \Phi^+ | M \otimes N | \Phi^+ \rangle = \frac{1}{2}\mathrm{tr}(MN^T)$ for Pauli operators:

$$E(A_0B_0) = \tfrac{1}{2}\mathrm{tr}(Z(Z + X)/\sqrt{2}) = \tfrac{1}{2\sqrt{2}}\mathrm{tr}(Z^2) = \tfrac{1}{\sqrt{2}}.$$

$$E(A_0B_1) = \tfrac{1}{2}\mathrm{tr}(Z(Z - X)/\sqrt{2}) = \tfrac{1}{\sqrt{2}}.$$

$$E(A_1B_0) = \tfrac{1}{2}\mathrm{tr}(X(Z + X)/\sqrt{2}) = \tfrac{1}{2\sqrt{2}}\mathrm{tr}(X^2) = \tfrac{1}{\sqrt{2}}.$$

$$E(A_1B_1) = \tfrac{1}{2}\mathrm{tr}(X(Z - X)/\sqrt{2}) = -\tfrac{1}{\sqrt{2}}.$$

Step 2: CHSH sum.

$$S = E(A_0B_0) + E(A_0B_1) + E(A_1B_0) - E(A_1B_1) = \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - (-\tfrac{1}{\sqrt{2}}) = \tfrac{4}{\sqrt{2}} = 2\sqrt{2}.$$

The Tsirelson maximum $2\sqrt{2} \approx 2.828$ is achieved exactly. The LHV bound 2 is violated by a margin of $2(\sqrt{2}-1) \approx 0.828$.

Step 3: Monte Carlo simulation. Draw N samples per setting. For setting (A_x, B_y), diagonalize $A_x \otimes B_y$ in the Bell state and sample outcomes $a, b \in \{\pm 1\}$ with probability

$$p(a, b|x, y) = \tfrac{1}{4}(1 + abE(A_xB_y)).$$

For (A_0, B_0): $p(+,+) = p(-,-) = (1+1/\sqrt{2})/4 \approx 0.427$, $p(+,-) = p(-,+) = (1-1/\sqrt{2})/4 \approx 0.073$. For (A_1, B_1): $p(+,+) = p(-,-) \approx 0.073$, $p(+,-) = p(-,+) \approx 0.427$.

With $N = 10^4$ samples per setting, the empirical CHSH estimate has standard error $\sim 4/\sqrt{N} = 0.04$, so $\hat{S} = 2.83 \pm 0.04$ — a 20σ violation of the LHV bound.

Step 4: Classical simulation fails. Any classical simulation that tries to reproduce these correlations using pre-generated shared randomness λ must satisfy $\hat{S} \leq 2$ by construction (see Exercise 1). No amount of classical cleverness can produce $\hat{S} > 2$; this is the content of Bell's theorem.

Experimental match. Loophole-free Bell tests (Hensen et al. 2015 with NV centers, Giustina et al. 2015 and Shalm et al. 2015 with entangled photons) all measured CHSH values consistent with $2\sqrt{2}$ to high statistical confidence. The world agrees with the quantum prediction, not the local-hidden-variable bound.

Cross-Links

- **Module 4.4:** CHSH operator and Bell states.
- **Module 6:** Interpretations module (Bell rules out local hidden variables; the survivors are precisely the non-local interpretations).
- **Statistics:** Bell's theorem is a constraint on conditional probability tables; CHSH is a no-go for hidden-variable causal models.
- **Quantum Information:** Device-independent cryptography.

Structural Compression

Invariant: The set of correlations achievable by local hidden-variable models is strictly smaller than the set achievable by quantum mechanics.

Mechanism: CHSH inequality ≤ 2 (LHV) vs $\leq 2\sqrt{2}$ (QM), experimentally verified to favor QM.

Cross-System Echo: In causal inference (Pearl), Bell-type inequalities are constraints on observable correlations imposed by latent-variable causal models. The quantum violation is the canonical example of the world refusing to be modeled by a classical causal graph.

Exercises

1. Verify the CHSH classical bound algebraically by enumerating the four cases for $B_0(\lambda) \pm B_1(\lambda)$.
2. Compute the CHSH expectation value for $|\Phi^+\rangle$ under the measurement choices given above.
3. Show that Tsirelson's bound $2\sqrt{2}$ cannot be exceeded by any quantum state.

4. State precisely which assumption fails in the local hidden-variable proof when applied to entangled quantum systems.
5. Argue why Bell's theorem makes quantum mechanics empirically distinguishable from classical physics in a way that *no* other quantum prediction does.

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Concepts: entanglement, conditional-probability, hypothesis-testing
Prerequisites: 4.4, 6.1
Enables: 10.2, 10.3

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