Kochen–Specker and Contextuality

Quantum Foundations · Module 10 — Foundations And Open Problems

Node 10.2

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Foundations and Open Problems **Node:** 10.2

Kochen–Specker (KS) is the second foundational no-go theorem of quantum mechanics, complementary to Bell. Where Bell's theorem rules out *local* hidden variables, KS rules out *non-contextual* hidden variables — even if non-locality is allowed. The distinction matters: Bell exploits the spacelike separation of two subsystems, but KS is a **single-system** theorem that needs no entanglement, no separation, and no statistics. It is a logical-algebraic obstruction: it is simply *impossible* to assign definite pre-existing values to all the observables of a quantum system of dimension $d \geq 3$ in a way that is consistent with the algebraic relations among them.

Formal Statement

Kochen–Specker theorem (1967). Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$. There is no function $v : \mathcal{O} \to \mathbb{R}$ on the set $\mathcal{O}$ of self-adjoint observables on $\mathcal{H}$ satisfying both:

1. **Spectrum rule.** For every observable A, $v(A)$ is an eigenvalue of A.
2. **Functional composition (non-contextuality).** For any commuting set $\{A_1, ..., A_n\}$ of observables and any function f such that $B = f(A_1, ..., A_n)$ is also an observable, $v(B) = f(v(A_1), ..., v(A_n))$.

In particular, if $A_1, ..., A_d$ are mutually commuting projectors summing to the identity (an orthonormal resolution of I), then exactly one of them must be assigned the value 1 and the rest must be assigned 0. KS shows that this assignment requirement is impossible: there exist finite sets of projectors that admit no consistent $\{0, 1\}$-coloring at all.

Sharper combinatorial version (Peres 1991, Cabello 1996): There exist explicit finite sets of n vectors in $\mathbb{C}^3$ or $\mathbb{C}^4$ such that no function v : vectors $\to \{0, 1\}$ satisfies the constraint "in every orthogonal triple (or quadruple), exactly one vector is assigned 1." The smallest known such set in $\mathbb{R}^3$ has 33 vectors (Peres); in $\mathbb{R}^4$, 18 vectors (Cabello).

Physical content: The value of a quantum observable cannot be a pre-existing property of the system; it can only be defined relative to a specific measurement *context* — the full set of commuting observables being measured alongside it.

Intuition

Classical physics allows you to ask "what is the x-component of the momentum *right now*, whether or not anyone is measuring it?" KS says: in quantum mechanics, if you try to build a hidden-variable theory in which every observable has a definite pre-measurement value, you run into a logical dead end. Not a statistical one — a purely combinatorial one. You try to color a finite set of vectors with 0s and 1s so that each basis has exactly one 1, and it can't be done.

The way out is: **the value of an observable depends on the context** — on which other commuting observables you are measuring it with. An observable A has one value when measured with $\{B, C\}$ and potentially a different value when measured with $\{B', C'\}$, even though A commutes with both sets. This is **contextuality**.

Bohmian mechanics escapes KS because it is explicitly contextual: the "hidden variable" (the particle position) determines outcomes of position measurements directly, but outcomes of other observables are determined by the full experimental setup, not by a pre-existing property of the particle alone.

Structural Role

KS sharpens and extends Bell in three ways:

1. **Dimension matters.** KS needs $d \geq 3$. For $d = 2$ (a single qubit), a non-contextual value assignment *does* exist (Gleason's theorem fails to bite in dimension 2). This is why KS is a genuinely "three-or-higher" phenomenon while Bell is a "two-party" phenomenon.
2. **No entanglement required.** KS is a single-system theorem. You don't need two separated labs.
3. **Logical, not statistical.** KS's impossibility is a coloring obstruction; it doesn't rely on violating inequalities by some amount. There are finite sets of vectors that *cannot* be colored, period.

Contextuality is now recognized as a genuine quantum resource:

- **Magic-state distillation** (Howard, Wallman, Veitch, Emerson 2014): contextuality is the structural resource that makes fault-tolerant quantum computing possible. Stabilizer operations alone are non-contextual (and classically simulable); contextuality of magic states is what enables universal quantum computation.
- **Contextuality as computational advantage:** parity-oblivious multiplexing, CSW correlations.
- **Spekkens toy model**: a non-contextual hidden-variable theory that reproduces *some* quantum phenomena, precisely those without contextuality. The phenomena it fails to reproduce are the ones that require contextuality.

In this module, KS sits between Bell (10.1) and PBR (10.3). Bell kills local realism; KS kills non-contextual realism; PBR kills ψ-epistemic realism. Together they carve out the remaining space of viable interpretations.

Mathematical Development

Peres–Mermin magic square

The cleanest finitary proof of contextuality (Mermin 1990, Peres 1990). Consider the 3×3 grid of two-qubit observables:

$X \otimes I$	$I \otimes X$	$X \otimes X$
$I \otimes Y$	$Y \otimes I$	$Y \otimes Y$
$X \otimes Y$	$Y \otimes X$	$Z \otimes Z$

Properties:

- Each row consists of three mutually commuting observables whose product is $+I$.
- Each column consists of three mutually commuting observables whose product is $+I$, **except** the third column, whose product is $-I$.

Now try to assign values $v \in \{+1, -1\}$ to each of the nine cells such that each row's product of values is $+1$ and each column's product is $+1, +1, -1$. Taking the product of all nine values two ways:

- By rows: $(+1)(+1)(+1) = +1$.

- By columns: $(+1)(+1)(-1) = -1$.

Contradiction. No such assignment exists. ■

The observables are all Pauli products on two qubits ($d = 4$), and each commuting row/column defines a measurement context. Contextuality shows up as: the "value" of $X \otimes X$ must be $+1$ when measured with $\{X \otimes I, I \otimes X\}$ (row 1) but -1 when measured with $\{Y \otimes Y, Z \otimes Z\}$ (column 3, which includes products forcing this sign).

Peres 33-vector proof in $\mathbb{R}^3$

For $d = 3$, take 33 specific unit vectors arranged in overlapping orthogonal triples. The coloring constraint is "exactly one 1 per triple." Peres showed a set of 33 vectors with this structure cannot be consistently colored; the constraint graph has no valid 2-coloring (it contains odd cycles of coloring forces). Cabello found an 18-vector set in $\mathbb{R}^4$ with the same property, which is the current smallest-known KS proof.

Connection to Gleason's theorem

Gleason's theorem (1957): in $d \geq 3$, the only functions μ on projectors satisfying $\mu(P) \in [0, 1]$ and additivity over orthogonal sets are of the form $\mu(P) = \mathrm{tr}(\rho P)$ for some density matrix ρ. KS can be read as a corollary: if in addition μ takes values in $\{0, 1\}$, then no such μ exists for $d \geq 3$, because density matrices don't have 0/1 spectra outside of $d = 1$ projections. Thus the very idea of a "$\{0, 1\}$-valued Gleason measure" is ruled out by the Gleason–KS combination.

State-independent vs state-dependent KS

The magic-square proof is **state-independent**: the contradiction arises from the algebra alone, not from any particular quantum state. It means *every* quantum state of the two-qubit system exhibits contextuality.

State-dependent KS proofs (Klyachko et al. 2008; the pentagon inequality) give inequalities that particular states violate. These are experimentally more convenient — you can verify them with a CHSH-like protocol.

Worked Example

Verifying Peres–Mermin in quantum mechanics

For the Pauli-product observables in the magic square, the products along each row and column are genuine operator identities. Check row 1:

$$(X \otimes I)(I \otimes X)(X \otimes X) = X \otimes X \cdot X \otimes X = I \otimes I = +I. \checkmark$$

Check column 3:

$$(X \otimes X)(Y \otimes Y)(Z \otimes Z) = (XYZ) \otimes (XYZ) = (iI) \otimes (iI) = -I \otimes I = -I. \checkmark$$

So the operator algebra is consistent: products of commuting observables yield exactly $+I$ along rows and along the first two columns, and $-I$ along the third. The contradiction arises only when one insists on assigning classical values to the individual observables.

Running the experiment

Mermin–Peres contextuality can be tested directly: prepare a two-qubit state, measure the three observables in a row, multiply outcomes, and check that the product is $+1$ with certainty (for the $+I$ rows) or -1 with certainty (for the $-I$ column). Every row and column's product is deterministic. The contradiction shows up as: there is no *single* value assignment $v(X \otimes X) \in \{+1, -1\}$ that makes row 1's product $+1$ *and* column 3's product -1, even though both row and column measurements are physically realizable.

Experimentally: this has been done in photonic, trapped-ion, and NMR systems (Kirchmair et al. 2009 in trapped ions). The observed products are as predicted, confirming the contextual structure.

Cross-Links

- **10.1 Bell's theorem** — the local case; KS is the contextual generalization that doesn't need spacelike separation.
- **10.3 PBR theorem** — another no-go on hidden-variable models, this time targeting ψ-epistemic theories.
- **6.4 Bohmian mechanics** — explicitly contextual, which is why it survives KS.
- **7.3 Quantum channels** — magic-state distillation uses contextuality as a resource.
- **2.2 Observables** — KS is a constraint on how observables can be assigned values.
- **Quantum logic** — the failure of distributivity in the lattice of quantum propositions is the logical shadow of KS.

Structural Compression

Invariant: No global function $v : \mathcal{O} \to \mathbb{R}$ can assign values to all observables of a $d \geq 3$ system consistently with the algebra.

Mechanism: Coloring obstruction on finite vector / observable sets. Peres–Mermin magic square; Peres 33-vector set; Cabello 18-vector set.

Cross-System Echo: Frustrated systems in statistical mechanics (no valid ground state assignment). Logical paradoxes in non-distributive lattices. Cohomological obstructions in sheaf theory (Abramsky–Brandenburger 2011 recast contextuality as a sheaf-theoretic obstruction).

Exercises

1. Verify the nine operator identities in the Peres–Mermin magic square: three row products are $+I$, two column products are $+I$, one column product is $-I$. Use the Pauli commutation and anti-commutation relations.

2. Show that in $d = 2$, a non-contextual $\{0, 1\}$-valued assignment to projectors *does* exist, by exhibiting one. Conclude that KS is a genuinely $d \geq 3$ phenomenon.

3. From the operator identities of the magic square, argue directly that no classical value assignment is consistent: take the product of all nine values two ways and derive the $+1 = -1$ contradiction.

4. Show that a non-contextual hidden-variable theory must assign definite values to all 18 vectors of Cabello's $\mathbb{R}^4$ proof, and sketch how the coloring constraints from orthogonal 4-tuples force a contradiction.

5. Explain why Bohmian mechanics is contextual, using the fact that the Bohmian particle position determines outcomes of position measurements only, and that outcomes of other observables are determined by the experimental setup.

6. Show that the magic-square test can be reformulated as a non-local game: two players share entanglement and are each asked a random row or column; they win if their answers are consistent. Compute the maximum classical winning probability and the quantum winning probability, and show that the latter is 1.

7. **Argue, structurally**, why contextuality is a stronger condition than non-locality — in particular, why every non-local hidden-variable theory fails KS but not every KS-violating theory is non-local.

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Concepts: eigenvalue-decomposition, hilbert-space
Prerequisites: 10.1, 2.2
Enables: 10.3

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