The PBR Theorem

Quantum Foundations · Module 10 — Foundations And Open Problems

Node 10.3

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Foundations and Open Problems **Node:** 10.3

The Pusey–Barrett–Rudolph (PBR) theorem is the third major no-go of quantum foundations, and the first one that targets the **ontological status of the wavefunction** directly. Bell (10.1) and Kochen–Specker (10.2) constrained the possible *values* assigned to observables. PBR constrains something more basic: whether the wavefunction ψ itself refers to physical reality or merely to an agent's knowledge of some deeper underlying state. Under a natural independence assumption, PBR rules out the latter view — the wavefunction must be part of the ontology.

Formal Statement

PBR theorem (Pusey, Barrett, Rudolph 2012). Consider a hidden-variable (ontological) model in which:

1. Each physical system has an underlying ontic state $\lambda \in \Lambda$.
2. Each quantum-state preparation $|\psi\rangle$ gives rise to a probability distribution $\mu_\psi(\lambda)$ over ontic states.
3. The outcome probabilities of any measurement M on the prepared system are determined by λ and M alone: $p(k|\psi, M) = \int \mu_\psi(\lambda)\, \xi_M(k|\lambda)\, d\lambda$.

Call the model ψ**-epistemic** if there exist two distinct pure states $|\psi\rangle, |\phi\rangle$ whose distributions μ_ψ, μ_ϕ have non-zero overlap on Λ — i.e. some ontic states λ are "compatible with both." Call it ψ**-ontic** otherwise: every pair of distinct pure states has disjoint distributions.

Preparation Independence Postulate (PIP). If two systems are prepared independently in states $|\psi_A\rangle$ and $|\phi_B\rangle$, the joint distribution of ontic states factors: $\mu_{\psi\otimes\phi}(\lambda_A, \lambda_B) = \mu_\psi(\lambda_A)\, \mu_\phi(\lambda_B)$.

Theorem. No ontological model satisfying PIP can be ψ-epistemic while reproducing the predictions of quantum mechanics.

Equivalently: if PIP holds and the quantum predictions are correct, then the wavefunction is ontic — it is part of the underlying physical state, not merely an agent's knowledge about it.

Intuition

A ψ-epistemic model says: the quantum state $|\psi\rangle$ is like a probability distribution in classical statistics. Two non-orthogonal pure states $|\psi\rangle$ and $|\phi\rangle$ are like two overlapping Gaussians — they cover different but intersecting regions of some underlying parameter space Λ. An ontic state λ in the intersection is compatible with both preparations; an experimenter who happens to prepare that λ is "really" in a state that either name would describe correctly.

If this picture were right, then the non-orthogonality of $|\psi\rangle$ and $|\phi\rangle$ (their non-zero inner product) would reflect only our ignorance about which preparation "really" occurred. The actual physics would be in λ, and the wavefunction would be derivative.

PBR ruins this picture. They construct a measurement on two independently prepared systems whose outcomes are **anti-distinguishable**: one of the four outcomes is guaranteed to never occur if the joint preparation was truly a specific product $|\psi\rangle \otimes |\phi\rangle$ for each of four possible combinations. But if the ontic states of the two systems are sampled independently from overlapping distributions μ_ψ, μ_ϕ, some ontic-state pairs will be in the intersection of both μ_ψ and μ_ϕ. The measurement apparatus doesn't "know" the label — it only sees (λ_A, λ_B) — so it has to output *something* on those joint ontic states. Whatever it outputs will, in one of the four combinations, be the forbidden outcome, contradicting the quantum prediction.

The only way to escape is to deny PIP: accept that independently prepared systems can have correlated ontic states. But that's a radical move — it means independence of preparation is a cognitive illusion, not a physical fact.

Structural Role

PBR lives on a different axis than Bell/KS:

- **Bell** constrains *locality* of hidden variables.
- **KS** constrains *non-contextuality* of value assignments.
- **PBR** constrains *ontological status of the wavefunction itself*.

Together these three theorems partition the space of possible interpretations:

Interpretation	Bell	KS	PBR
Local HV	fails	fails	—
Non-contextual HV	varies	fails	—
ψ-epistemic + PIP	varies	varies	**fails**
ψ-ontic (Bohm, MWI, GRW, Copenhagen)	passes	passes	passes
QBism (rejects PIP + ontic reading)	N/A	N/A	escapes by rejecting PIP
Superdeterminism (rejects measurement independence)	escapes all three	—	—

The interpretations that survive all three no-gos are exactly the ψ-ontic ones (Bohmian, MWI, GRW, Copenhagen) plus those (QBism, superdeterminism, relational QM) that reject one of the background assumptions.

PBR is also the answer to a long-standing question: is the wavefunction "knowledge" or "reality"? PBR's answer: under PIP, it's reality.

Mathematical Development

The PBR protocol

Consider two qubits, each independently prepared in one of two states: $|0\rangle$ or $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The four possible preparations are

$$|00\rangle, \quad |0+\rangle, \quad |+0\rangle, \quad |++\rangle.$$

These four product states are not mutually orthogonal (they pairwise overlap by 1/2 or 1/4), but they have a remarkable property: there exists an entangled 4-outcome orthogonal measurement $\{|\xi_1\rangle, |\xi_2\rangle, |\xi_3\rangle, |\xi_4\rangle\}$ such that

- $\langle\xi_1|00\rangle = 0$
- $\langle\xi_2|0+\rangle = 0$
- $\langle\xi_3|+0\rangle = 0$
- $\langle\xi_4|++\rangle = 0$

That is, each of the four quantum outcomes is "forbidden" for exactly one of the four preparations. In quantum mechanics this is a concrete fact about the 4-outcome measurement in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The PBR argument

Suppose we have a ψ-epistemic ontological model with PIP. Then the distributions $\mu_0(\lambda)$ and $\mu_+(\lambda)$ have non-zero overlap — there is a subset $\Lambda^* \subset \Lambda$ with $\mu_0(\Lambda^*), \mu_+(\Lambda^*) > 0$.

By PIP, the joint distribution on pairs (λ_A, λ_B) for a product preparation is the product of the marginals. So for each of the four preparations, the joint distribution assigns positive probability to the region $\Lambda^* \times \Lambda^*$.

Now consider a pair $(\lambda_A, \lambda_B) \in \Lambda^* \times \Lambda^*$. This pair is "compatible" with all four preparations. So the measurement apparatus, seeing only this pair and the measurement setting, must output one of the four outcomes $\{1, 2, 3, 4\}$. Say it outputs outcome k with probability $\xi_M(k|\lambda_A, \lambda_B)$.

By consistency with quantum mechanics, the outcome probabilities on this pair, averaged over the distribution, must reproduce the quantum prediction of zero probability for the forbidden outcome. But the pair is in $\Lambda^* \times \Lambda^*$ for *all four* preparations, so we need $\xi_M(1) = \xi_M(2) = \xi_M(3) = \xi_M(4) = 0$ on this pair. That's impossible — they must sum to 1. ■

Refinements and extensions

- **Multiple copies (Moseley 2013; Colbeck–Renner 2017):** PBR's original proof used 2 copies; subsequent work showed n copies can give tighter bounds or deal with noise.
- **Experimental tests:** Nigg et al. (2016) tested a PBR-like protocol with trapped ions. Results consistent with quantum predictions; ψ-epistemic models with PIP ruled out to high confidence.
- **Hardy's ψ-ontic theorem (2013):** A related no-go reaching the same conclusion through a simpler but less general argument.

The PBR measurement explicitly

One valid choice of the four measurement vectors is:

$$|\xi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\xi_2\rangle = \frac{1}{\sqrt{2}}(|0-\rangle + |1+\rangle)$$
$$|\xi_3\rangle = \frac{1}{\sqrt{2}}(|-0\rangle + |+1\rangle)$$
$$|\xi_4\rangle = \frac{1}{2}(|0-\rangle - |1+\rangle - |-0\rangle - |+1\rangle + ...)$$

(The exact forms are set by orthogonality and the anti-distinguishability requirements; several equivalent bases exist.) Direct computation verifies $\langle\xi_k|\text{preparation}_k\rangle = 0$ for each k.

Worked Example

Verifying anti-distinguishability

Take the first constraint: $\langle \xi_1 | 00 \rangle = 0$. For $|\xi_1\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$:

$$\langle \xi_1 | 00 \rangle = \frac{1}{\sqrt{2}}(\langle 01|00\rangle + \langle 10|00\rangle) = \frac{1}{\sqrt{2}}(0+0) = 0. \checkmark$$

So if the preparation is $|00\rangle$, outcome 1 is forbidden in quantum mechanics. Similar computations for the other three preparations — each has exactly one forbidden outcome.

The ontological contradiction

Suppose $\mu_0(\lambda)$ and $\mu_+(\lambda)$ have overlap region Λ^* with $\mu_0(\Lambda^*) = \mu_+(\Lambda^*) = q > 0$. Then for each of the four product preparations, the probability that the joint ontic state lies in $\Lambda^* \times \Lambda^*$ is $q^2 > 0$.

If the joint ontic state is in this region, the measurement apparatus cannot distinguish which of the four preparations produced it. Its output distribution $\xi_M(k|\lambda_A, \lambda_B)$ is a single function of (λ_A, λ_B), not of the label. But quantum mechanics demands:

- Outcome 1 forbidden if prep was $|00\rangle$
- Outcome 2 forbidden if prep was $|0+\rangle$
- Outcome 3 forbidden if prep was $|+0\rangle$
- Outcome 4 forbidden if prep was $|++\rangle$

On the q^2 fraction of joint ontic states in $\Lambda^* \times \Lambda^*$, the same function ξ_M has to output outcome 1 with probability 0 (because preparation could have been $|00\rangle$) and outcome 2 with probability 0 (because it could have been $|0+\rangle$) and similarly for 3 and 4 — all four outcomes with probability 0. Sum of probabilities is 0, not 1. Contradiction.

Therefore $q = 0$: the distributions μ_0 and μ_+ have zero overlap, and the model is ψ-ontic.

Cross-Links

- **10.1 Bell's theorem** — constrains locality of hidden variables.
- **10.2 Kochen–Specker** — constrains non-contextuality of value assignments.
- **6.6 Interpretations empirical equivalence** — this is the no-go that rules out ψ-epistemic + PIP.
- **6.5 QBism** — rejects PIP, so it evades PBR.
- **4.4 Entanglement** — the four-outcome PBR measurement is genuinely entangled.
- **Ontological models framework** (Spekkens 2005) — formal setting in which PBR is stated.

Structural Compression

Invariant: ψ-epistemic + preparation independence is incompatible with quantum mechanics.

Mechanism: Anti-distinguishability of four product states by an entangled 4-outcome measurement. Forces the ontic-state distributions of distinct pure states to be disjoint.

Cross-System Echo: Statistical inference cannot distinguish overlapping hypotheses with certainty, but quantum anti-distinguishability *does* give certainty-of-exclusion for specific outcomes. The quantum world is more decisive than classical statistics, and PBR shows that this decisiveness is structurally incompatible with "ψ is knowledge."

Exercises

1. Verify directly that $\langle \xi_k | \text{preparation}_k \rangle = 0$ for each of the four PBR preparations and the four measurement outcomes, given an explicit (4, 4)-pair of orthonormal bases satisfying the anti-distinguishability constraint.
2. Show that anti-distinguishability of n states requires an n-outcome measurement where each outcome is orthogonal to one state, and that in general this is only possible when the states are sufficiently "spread out" in Hilbert space.
3. Construct a ψ-epistemic toy model (Spekkens 2005 is the standard reference) for a single qubit, and show explicitly that it reproduces single-qubit quantum predictions. Then show that when extended to two qubits via a product rule, it *cannot* reproduce the PBR anti-distinguishing measurement.
4. Explain why QBism, which treats $|\psi\rangle$ as an agent's personal probability assignment, is not refuted by PBR. Identify the specific assumption QBism rejects.
5. Show that if one allows arbitrarily correlated hidden states between independent preparations (denying PIP), a trivial ψ-epistemic model reproducing all quantum predictions exists. Comment on why this escape is considered unphysical by most interpreters.
6. A retrocausal model allows the hidden-variable distribution to depend on the future measurement setting. Show that this evades PBR, and explain the cost in terms of time-symmetry.
7. **Argue, structurally**, why PBR is a no-go along a genuinely different axis than Bell and KS. Identify the assumption each theorem attacks, and show that the three assumptions are logically independent (one can construct toy models that fail any one but pass the other two).

Quantum Foundations · Module 10 — Foundations And Open Problems · Node 10.3
Concepts: probability-space, hypothesis-testing
Prerequisites: 10.1, 6.6

hbar.university — own.your.cognition