The State Postulate

Quantum Foundations · Module 2 — Postulates

Node 2.1

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Postulates of Quantum Mechanics **Node:** 2.1

This is the first injection of physical content into the linear-algebraic substrate of Module 1. Up to this point we have built a complex Hilbert space — a mathematical object. The state postulate identifies a specific subset of that object with the physical states of a quantum system.

Formal Statement

To every isolated physical system there is associated a complex separable Hilbert space $\mathcal{H}$, called the **state space** of the system.

The states of the system are represented by **rays** in $\mathcal{H}$: equivalence classes

$$[|\psi\rangle] = \{\alpha|\psi\rangle : \alpha \in \mathbb{C}, \alpha \neq 0\}$$

with $|\psi\rangle \in \mathcal{H}$ nonzero.

By convention, one fixes a representative of unit norm:

$$\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle} = 1.$$

Even after normalization, the **global phase** is not determined: $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same physical state for any $\theta \in \mathbb{R}$.

Intuition

A quantum state is not a vector. It is a **direction** in Hilbert space — a one-dimensional subspace, with magnitude and global phase stripped away.

The reason is the Born rule (Node 2.3): all measurable predictions take the form $|\langle\phi|\psi\rangle|^2$, and this quantity is invariant under $|\psi\rangle \mapsto e^{i\theta}|\psi\rangle$. Two vectors that differ only by an overall complex factor produce identical measurement statistics, so they cannot represent physically distinct situations.

What survives the quotient by global phase is the **projective Hilbert space** $\mathbb{P}\mathcal{H}$. This — not $\mathcal{H}$ itself — is the true state space of a quantum system.

Crucial caveat: the equivalence is by *global* phase only. **Relative** phases between components of a superposition are physical and observable (interference depends on them).

Structural Role

This postulate fixes the ontology of the rest of the course:

- **Observables** (Node 2.2) will be Hermitian operators on $\mathcal{H}$.
- **Measurement outcomes** (Node 2.3) will be eigenvalues, with probabilities determined by inner products.
- **Time evolution** (Node 2.4) will be unitary, hence ray-preserving.
- **Composite systems** (Node 2.5) will live in tensor products of state spaces.
- **Mixed states** (Node 5.1) will generalize rays to density operators when statistical ensembles are introduced.

Every other postulate is calibrated against this one.

Mathematical Development

The projective Hilbert space

The projective Hilbert space is the quotient

$$\mathbb{P}\mathcal{H} = (\mathcal{H} \setminus \{0\})/\sim$$

where $|\psi\rangle \sim |\phi\rangle \iff |\phi\rangle = \alpha|\psi\rangle$ for some $\alpha \in \mathbb{C}^*$.

For $\mathcal{H} = \mathbb{C}^{n+1}$, this is the complex projective space $\mathbb{CP}^n$. For the qubit, $\mathcal{H} = \mathbb{C}^2$ and $\mathbb{CP}^1$ is topologically the 2-sphere — the **Bloch sphere**.

Why rays, not vectors

Suppose two states $|\psi\rangle, e^{i\theta}|\psi\rangle$ were physically distinct. Then there would have to exist some measurement procedure that distinguishes them. But by the Born rule (anticipating 2.3), the probability of any outcome $|\phi\rangle$ is

$$|\langle\phi|e^{i\theta}\psi\rangle|^2 = |e^{i\theta}|^2\,|\langle\phi|\psi\rangle|^2 = |\langle\phi|\psi\rangle|^2.$$

The two vectors are operationally indistinguishable, so the postulate identifies them.

Worked Example

A qubit on the Bloch sphere

A general qubit state can be written

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\varphi}\sin(\theta/2)\,|1\rangle$$

with $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$. This parameterization eliminates global phase by convention (the coefficient of $|0\rangle$ is real and non-negative).

The map $|\psi\rangle \mapsto (\theta,\varphi)$ sends qubit states bijectively onto the unit sphere $S^2 \subset \mathbb{R}^3$ — the Bloch sphere.

Key facts:

- Each point on the sphere is one quantum state (one ray).
- Antipodal points are orthogonal vectors in $\mathcal{H}$ (e.g., $|0\rangle$ and $|1\rangle$ are at the north and south poles).

- The interior of the ball corresponds to mixed states (introduced in Node 5.1).

Cross-Links

- **Linear Algebra:** Vector Spaces, Quotient Spaces.
- **Statistics:** Classical state = point in a sample space. Quantum state = ray in a Hilbert space. The structural difference is what generates all subsequent quantum-vs-classical distinctions.
- **VQA:** Parameterized quantum circuits prepare rays $|\psi(\theta)\rangle$; the cost function is a real-valued function on the projective Hilbert space.
- **Quantum Information:** Qubit states, qudit states, computational basis.

Structural Compression

Invariant: Physical states are equivalence classes under global phase, not raw Hilbert vectors.

Mechanism: Quotient of $\mathcal{H} \setminus \{0\}$ by complex scalar multiplication.

Cross-System Echo: Probability distributions in classical statistics live on the simplex (a real convex set); quantum states live on the projective Hilbert space (a complex projective manifold). The non-convexity of pure states is the structural marker that quantum mechanics is irreducibly different.

Exercises

1. Show that the relation $|\psi\rangle \sim \alpha|\psi\rangle$ (for $\alpha \in \mathbb{C}^*$) is an equivalence relation on $\mathcal{H} \setminus \{0\}$.
2. Verify that the Bloch sphere parameterization $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$ covers every qubit state exactly once.
3. Show that two qubit states are orthogonal in $\mathbb{C}^2$ if and only if their Bloch vectors are antipodal.
4. Argue why the relative phase between $|0\rangle$ and $|1\rangle$ in a superposition $\alpha|0\rangle + \beta|1\rangle$ is physical, even though the global phase is not.

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Concepts: hilbert-space, state-space, superposition
Prerequisites: 1.3, 1.4
Enables: 2.2, 2.3, 2.4, 2.5

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