The Observable Postulate

Quantum Foundations · Module 2 — Postulates

Node 2.2

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Postulates of Quantum Mechanics **Node:** 2.2

The state postulate (2.1) said *what a state is*. This postulate says *what we are allowed to ask about a state*. Together they fix the ontology of measurement before the Born rule (2.3) supplies the statistics.

Formal Statement

To every measurable physical quantity (an **observable**) of a quantum system is associated a self-adjoint (Hermitian) operator on the system's Hilbert space:

$$A : \mathcal{H} \to \mathcal{H}, \qquad A = A^\dagger.$$

The possible outcomes of a measurement of the observable are the elements of the **spectrum** of A:

- For a bounded operator in finite dimensions, the spectrum is the set of real eigenvalues $\{\lambda_i\}$ of the spectral decomposition $A = \sum_i \lambda_i P_i$ (Node 1.6).
- For an unbounded self-adjoint operator on an infinite-dimensional Hilbert space, the spectrum is a closed subset of $\mathbb{R}$ described by a projection-valued measure E_A, and the observable is realized as $A = \int \lambda \, dE_A(\lambda)$.

In finite dimensions, Hermiticity and self-adjointness are the same condition. In infinite dimensions they differ: self-adjointness requires both symmetry ($\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ on the common domain) and equality of domains $\mathrm{dom}(A) = \mathrm{dom}(A^\dagger)$. Only genuinely self-adjoint operators admit a spectral measure, and only these are legitimate observables.

Intuition

An observable is encoded by an operator because measurement in quantum mechanics is non-commutative: the result of measuring "A then B" differs in general from "B then A", and operators are the mathematical objects whose composition is non-commutative.

Hermiticity is chosen because:

1. **Measurement outcomes are real numbers.** The eigenvalues of a Hermitian operator are real (Node 1.6). Complex eigenvalues would correspond to unobservable quantities.
2. **Expectation values are real.** For any state $|\psi\rangle$, the number $\langle \psi | A | \psi \rangle$ is real if and only if A is Hermitian. Reality of expectation values is a necessary condition for A to represent an empirical mean.
3. **Spectral decomposition exists.** Hermiticity is the minimal algebraic condition under which the spectral theorem guarantees an orthogonal decomposition of Hilbert space into eigenspaces — which is what the Born rule (2.3) and projective measurement (3.1) consume.

Non-Hermitian operators still appear in quantum mechanics — raising and lowering operators, annihilation operators, Kraus operators — but they describe *transformations between states*, not measurable quantities. Amplitudes, not outcomes.

Structural Role

The observable postulate is the hinge that turns the spectral theorem (1.6) into physics.

- It dictates that all operators appearing in the Born rule (2.3) come equipped with real spectra and orthogonal eigenspaces.
- It licenses the correspondence "operator eigenvalues $\leftrightarrow$ possible outcomes."
- It sets up the notion of **compatible observables**: observables whose operators commute can be measured jointly; those whose operators do not cannot (Node 3.5).
- It is the structural origin of uncertainty relations — which are statements about pairs of non-commuting observables.

Downstream, it also sets up the Hamiltonian as the distinguished observable that generates time evolution (Node 2.4).

Mathematical Development

Expectation and variance

For a state $|\psi\rangle$ and an observable A, the **expectation value** is

$$\langle A\rangle_\psi := \langle\psi|A|\psi\rangle.$$

With spectral decomposition $A = \sum_i \lambda_i P_i$,

$$\langle A\rangle_\psi = \sum_i \lambda_i \langle\psi|P_i|\psi\rangle = \sum_i \lambda_i \, \|P_i|\psi\rangle\|^2.$$

The right-hand side is a weighted sum of eigenvalues with non-negative weights that sum to one — a genuine probabilistic expectation, as the Born rule (2.3) will confirm.

The **variance** of A in $|\psi\rangle$ is

$$(\Delta A)^2_\psi := \langle A^2\rangle_\psi - \langle A\rangle^2_\psi = \langle\psi|(A - \langle A\rangle_\psi I)^2|\psi\rangle.$$

A state is an **eigenstate** of A if and only if its variance vanishes: the outcome is deterministic.

Commuting observables and joint measurability

Two self-adjoint operators A, B are **compatible** if they commute: $[A, B] = 0$.

Theorem. Two Hermitian operators on a finite-dimensional Hilbert space commute if and only if they are simultaneously diagonalizable in a common orthonormal basis.

The eigenvectors of such a common diagonalization are labelled by joint eigenvalue pairs (λ, μ), and each such pair is a possible outcome of a simultaneous measurement of A and B. Non-commuting observables have no common eigenbasis and cannot be jointly measured with definite outcomes — the structural origin of quantum indeterminacy.

Robertson uncertainty relation

For self-adjoint A and B and any state $|\psi\rangle$,

$$\Delta A \cdot \Delta B \geq \tfrac{1}{2}|\langle [A,B]\rangle_\psi|.$$

The commutator is the algebraic origin of the uncertainty lower bound. Node 3.5 will revisit this; here we note only that it is a direct consequence of Cauchy–Schwarz applied to the state-dependent inner product.

Functional calculus

Because observables are Hermitian, any real-valued function of an observable is again an observable:

$$f(A) := \sum_i f(\lambda_i) P_i \quad \text{(for real } f\text{)}.$$

This is used constantly: A^2, $e^{-\beta A}$, and so on. The functional calculus is what makes "expectation of the square of A" a well-defined observable.

Worked Example

Pauli operators as qubit observables

The three Pauli operators X, Y, Z (Node 1.5) are Hermitian, have spectrum $\{+1, -1\}$, and constitute the basic observables of a qubit.

- **Measuring Z:** outcomes ± 1 corresponding to eigenvectors $|0\rangle, |1\rangle$ — the computational-basis measurement.
- **Measuring X:** outcomes ± 1 corresponding to eigenvectors $|+\rangle, |-\rangle$ — the Hadamard-basis measurement.
- **Measuring Y:** outcomes ± 1 corresponding to eigenvectors $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$.

Non-commutativity. Since $[X, Z] = -2iY \neq 0$, X and Z are not jointly measurable. There is no state with simultaneously definite values of both. The Robertson uncertainty relation gives

$$\Delta X \cdot \Delta Z \geq |\langle Y\rangle_\psi|,$$

which is saturated when $|\psi\rangle$ is a Y-eigenstate.

Position and momentum on $L^2(\mathbb{R})$

For a particle on a line, $\mathcal{H} = L^2(\mathbb{R})$. The position and momentum observables are

$$(X\psi)(x) = x\psi(x), \qquad (P\psi)(x) = -i\hbar\frac{d\psi}{dx}.$$

Both are unbounded self-adjoint operators, defined on dense domains. Their spectra are the entire real line $\sigma(X) = \sigma(P) = \mathbb{R}$; neither has normalizable eigenvectors, so "position eigenstates" $\delta(x - x_0)$ and "momentum eigenstates" $e^{ipx/\hbar}$ are formal distributions, not Hilbert-space vectors (Node 1.3).

The canonical commutation relation

$$[X, P] = i\hbar I$$

together with Robertson's inequality gives the Heisenberg uncertainty relation

$$\Delta X \cdot \Delta P \geq \frac{\hbar}{2}.$$

Number operator on Fock space

For a single bosonic mode, the Hilbert space is $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathbb{C}|n\rangle$ — the Fock space. The **number operator** N is defined by $N|n\rangle = n|n\rangle$, with spectrum $\{0, 1, 2, \ldots\}$. It is Hermitian and unbounded. The Hamiltonian of the harmonic oscillator $H = \hbar\omega(N + \frac{1}{2})$ is a real-linear function of N; its eigenvalues are the quantized energy levels.

Cross-Links

- **Linear Algebra:** Self-adjoint operators and the spectral theorem.
- **Statistics:** Random variables are the classical analogue of observables; the structural difference is that classical random variables commute (they are functions on a common sample space) and quantum observables need not.
- **VQA:** The cost-defining Hamiltonian is an observable; variational estimates are expectation values of this observable in parameterized states.
- **Quantum Information:** Measurement bases correspond to choices of observable; all single-qubit measurements reduce to rotations of Z by appropriate unitaries.
- **Quantum Thermodynamics:** Energy is the observable represented by the Hamiltonian; entropy is a function of the density operator, not itself an observable.

Structural Compression

Invariant: Observables are self-adjoint operators; their spectra are real sets of possible outcomes.

Mechanism: Hermiticity guarantees real eigenvalues, real expectation values, and orthogonal eigenspace decomposition — all three requirements of a physical measurement.

Cross-System Echo: Classical random variables on a common sample space are all jointly measurable because they are functions on the same underlying set. Quantum observables live as operators, which need not commute, and their failure to commute is exactly what makes quantum probability a genuine generalization of classical probability rather than an elaborate special case.

Exercises

1. Prove that the expectation value $\langle\psi|A|\psi\rangle$ is real if and only if $A = A^\dagger$.
2. Show that a state $|\psi\rangle$ has zero variance $\Delta A = 0$ if and only if $|\psi\rangle$ is an eigenstate of A.
3. Prove the Robertson uncertainty relation $\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle[A, B]\rangle_\psi|$. (Hint: apply Cauchy–Schwarz to the vectors $(A - \langle A\rangle)|\psi\rangle$ and $(B - \langle B\rangle)|\psi\rangle$.)
4. Compute $\langle X\rangle, \langle Z\rangle, \Delta X, \Delta Z$ for the qubit state $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$. Verify that $\Delta X \cdot \Delta Z \geq |\langle Y\rangle|$.
5. Show that two Hermitian operators on a finite-dimensional Hilbert space commute if and only if they admit a common orthonormal eigenbasis.
6. Verify the canonical commutation relation $[X, P] = i\hbar I$ by direct calculation on a smooth test function $\psi(x) \in L^2(\mathbb{R})$.
7. Give an example of a non-Hermitian operator used in quantum mechanics, and explain what role it plays. Argue why it is not itself an observable.

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Concepts: spectral-analysis, linear-operators, eigenvalue-decomposition
Prerequisites: 1.5, 1.6, 2.1
Enables: 2.3, 3.1

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