The Born Rule

Quantum Foundations · Module 2 — Postulates

Node 2.3

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Postulates of Quantum Mechanics **Node:** 2.3

The Born rule is the bridge from the quantum formalism to empirical reality. It is the only place where probabilities enter the postulates — and the only place where the formalism contacts measurement statistics. Remove it and everything in Modules 1 and 2 remains mathematically consistent and physically inert.

Formal Statement

Let A be an observable with spectral decomposition

$$A = \sum_i \lambda_i P_i,$$

and let $|\psi\rangle \in \mathcal{H}$ be a normalized state ($\|\psi\| = 1$). When A is measured:

1. **Outcome probabilities.** The probability of obtaining outcome λ_i is

$$p(\lambda_i) = \langle\psi|P_i|\psi\rangle = \|P_i|\psi\rangle\|^2.$$

2. **Post-measurement state (projection postulate).** Conditional on obtaining outcome λ_i, the state immediately after measurement is

$$|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{p(\lambda_i)}}.$$

The post-measurement state lies entirely in the eigenspace $\mathrm{ran}(P_i)$, so an immediately repeated measurement of A yields the same outcome with certainty.

If λ_i is non-degenerate with eigenvector $|\lambda_i\rangle$, the rule specializes to

$$p(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2, \qquad |\psi'\rangle = |\lambda_i\rangle.$$

The first formula — squared amplitude equals probability — is the colloquial "Born rule." The projection step is sometimes called the "collapse postulate"; its interpretive status is revisited in Module 6.

Intuition

A state $|\psi\rangle$ is a superposition of eigenstates of whichever observable you choose to measure. The Born rule reads off, from the amplitude of each eigenstate component, the probability that the measurement outcome will be the corresponding eigenvalue. "The state is α parts eigenstate-1 and β parts eigenstate-2" translates to "outcome 1 occurs with probability $|\alpha|^2$, outcome 2 with probability $|\beta|^2$."

The squared modulus is essential. It is what discards global phase (making physical states rays, per Node 2.1), preserves total probability (via the resolution of identity), and aligns complex amplitudes with real probabilities.

The projection postulate is the statement that, having observed a particular outcome, the state is now compatible with that outcome — it has been projected into the corresponding eigenspace. This is the single step in quantum mechanics that is not unitary, and it is the source of the measurement problem (Module 6).

Structural Role

The Born rule is the empirical interface of the theory. Every laboratory number that has ever been measured in a quantum experiment — every photon count, every neutron polarization, every qubit readout — is a sample from a Born-rule probability distribution.

- It consumes the spectral decomposition of observables (Node 1.6, 2.2).
- It is the foundation of projective measurement (Node 3.1), POVMs (Node 3.2), and weak measurement (Node 3.5).
- It is the ingredient that extends to density matrices (Node 5.1) via $p(\lambda_i) = \mathrm{Tr}(P_i\rho)$.
- It is what variational quantum algorithms (VQA course) sample from: every estimate of $\langle\psi(\theta)|H|\psi(\theta)\rangle$ is a finite-shot average over Born-rule samples.

Removing the Born rule leaves the rest of the postulates mutually consistent but empirically silent.

Mathematical Development

Total probability sums to one

From the resolution of identity $\sum_i P_i = I$ (spectral theorem, Node 1.6):

$$\sum_i p(\lambda_i) = \sum_i \langle\psi|P_i|\psi\rangle = \langle\psi| \left(\sum_i P_i\right) |\psi\rangle = \langle\psi|I|\psi\rangle = \|\psi\|^2 = 1.$$

This is why normalization matters and why the spectral theorem is the structural prerequisite. Total probability is conserved as an algebraic identity.

Expectation value

The expectation value of an observable equals the Born-weighted mean of its eigenvalues:

$$\langle A\rangle_\psi = \sum_i \lambda_i p(\lambda_i) = \sum_i \lambda_i \langle\psi|P_i|\psi\rangle = \langle\psi|A|\psi\rangle.$$

The first and last expressions are equal by the spectral decomposition. This identification — "expectation value equals $\langle\psi|A|\psi\rangle$" — is a *theorem*, not an additional postulate, once the Born rule and the spectral theorem are in place.

Variance and uncertainty

The variance of A in the state $|\psi\rangle$ is

$$(\Delta A)^2 = \sum_i (\lambda_i - \langle A\rangle)^2 p(\lambda_i) = \langle A^2\rangle - \langle A\rangle^2 = \langle\psi|(A - \langle A\rangle I)^2|\psi\rangle.$$

Zero variance $\iff$ $|\psi\rangle$ is an eigenstate of A $\iff$ the outcome is deterministic.

Degenerate eigenvalues

When λ_i has an eigenspace of dimension > 1, P_i is a projection onto that subspace. The outcome probability $\langle\psi|P_i|\psi\rangle$ depends only on the projection of $|\psi\rangle$ onto the eigenspace, not on the internal direction. The post-measurement state is $P_i|\psi\rangle/\sqrt{p(\lambda_i)}$, which is generally not an eigenvector of A alone — it is merely an element of the correct eigenspace.

Sequential measurements

Consider measuring A, obtaining λ_i, and then measuring $B = \sum_j \mu_j Q_j$. The joint probability is

$$p(\lambda_i, \mu_j) = \langle\psi|P_i Q_j P_i|\psi\rangle,$$

which in general is *not* the classical joint probability $p(\lambda_i)p(\mu_j|\lambda_i)$ with commuting conditionals. If $[A, B] = 0$, the projections commute and sequential measurements behave classically. If not, the order of measurement matters — the structural origin of quantum contextuality (Node 10.2).

Worked Example

Pauli Z on $|+\rangle$

Prepare the qubit state

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

and measure the observable Z, with spectral decomposition $Z = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$.

Outcome probabilities:

$$p(+1) = |\langle 0|+\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}, \qquad p(-1) = |\langle 1|+\rangle|^2 = \tfrac{1}{2}.$$

Post-measurement states: if the outcome is $+1$, the state collapses to $|0\rangle$; if -1, to $|1\rangle$.

Expectation value:

$$\langle Z\rangle_+ = (+1)\cdot\tfrac{1}{2} + (-1)\cdot\tfrac{1}{2} = 0 = \langle +|Z|+\rangle.$$

Variance:

$$(\Delta Z)^2_+ = \langle Z^2\rangle_+ - \langle Z\rangle^2_+ = 1 - 0 = 1.$$

The state $|+\rangle$ is maximally uncertain in Z, minimally uncertain (variance zero) in X — the tradeoff enforced by non-commutativity.

Repeated measurement

Prepare $|+\rangle$, measure Z, suppose the outcome is $+1$. The state is now $|0\rangle$. Measure Z again: the outcome is $+1$ with probability $|\langle 0|0\rangle|^2 = 1$. Deterministic repetition — the structural content of the projection postulate.

If instead, after measuring Z and obtaining $+1$, we measure X, we get ± 1 with probability $|\langle \pm|0\rangle|^2 = \frac{1}{2}$. The Z measurement has erased the state's X information.

Cross-Links

- **Linear Algebra:** The spectral theorem supplies $\sum_i P_i = I$, which is the algebraic source of "probabilities sum to one."
- **Statistics:** Probability measures on outcome spaces are the classical analogue. The Born rule is a non-commutative generalization — the probability space is state-dependent and observable-dependent.
- **VQA:** Every cost-function estimate is a finite-shot sample from a Born distribution, introducing shot-noise as a fundamental error source.
- **Quantum Information:** Distinguishability of quantum states is bounded by Born-rule overlap $|\langle \phi|\psi\rangle|^2$; non-orthogonal states are not perfectly distinguishable.
- **Module 5 (Decoherence):** The density-matrix form $p(\lambda_i) = \mathrm{Tr}(P_i \rho)$ generalizes the pure-state Born rule to statistical mixtures.
- **Module 6 (Interpretations):** The projection postulate is the single non-unitary step in the postulates; its interpretive status is the measurement problem.

Structural Compression

Invariant: Squared amplitudes are probabilities, and spectral projections resolve the identity.

Mechanism: $p(\lambda_i) = \langle \psi|P_i|\psi\rangle$, with total probability enforced by $\sum_i P_i = I$.

Cross-System Echo: Classical expectation is $\int X\, dP$ with dP a probability measure on a sample space. Quantum expectation is $\langle \psi|A|\psi\rangle$ with the "measure" depending on both the state and the observable. The structural role is identical; the non-commutativity is what is new.

Exercises

1. Verify that the Born rule gives total probability one by direct computation: $\sum_i \langle \psi|P_i|\psi\rangle = 1$ for any unit vector $|\psi\rangle$.
2. Derive the identity $\langle A\rangle_\psi = \sum_i \lambda_i p(\lambda_i)$ from the Born rule and the spectral decomposition.
3. For $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$, compute the Born probabilities for a Z-measurement and for an X-measurement. Explain why the answer for Z depends only on θ while the answer for X depends on both θ and φ.
4. Prepare $|0\rangle$ and measure X. What are the probabilities of each outcome and the post-measurement state in each case?
5. Show that if $[A, B] = 0$, then sequential measurements of A and B produce the same statistics as a single joint measurement (in a common eigenbasis). Show by example that this fails when $[A, B] \neq 0$.
6. Derive the generalization $p(\lambda_i) = \mathrm{Tr}(P_i \rho)$ of the Born rule from the pure-state version, assuming $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ is a statistical mixture of pure states. (This previews Node 5.1.)
7. The Born rule's squared modulus is sometimes called "the mysterious quadratic." Write a short paragraph arguing, structurally, why linear amplitudes ($p \propto |\alpha|$) and cubic amplitudes ($p \propto |\alpha|^3$) are both incompatible with the inner-product structure of Hilbert space.

Quantum Foundations · Module 2 — Postulates · Node 2.3
Concepts: born-rule, expectation-value, probability-space
Prerequisites: 2.1, 2.2, 1.6
Enables: 3.1, 3.2

hbar.university — own.your.cognition