Composite Systems and Tensor Products

Quantum Foundations · Module 2 — Postulates

Node 2.5

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Postulates of Quantum Mechanics **Node:** 2.5

This postulate is the structural origin of entanglement. The tensor product is *not* the Cartesian product, and that single distinction is responsible for everything quantum mechanics has that classical mechanics does not — the exponential state-space growth, Bell nonlocality, quantum computational speedups, and decoherence. Module 4 is devoted to exploring the consequences; here we lay down the structure.

Formal Statement

If a quantum system AB consists of two subsystems A and B with state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, then the state space of the composite system is the **tensor product**

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

Given orthonormal bases $\{|i\rangle_A\}_{i=1}^{d_A}$ for $\mathcal{H}_A$ and $\{|j\rangle_B\}_{j=1}^{d_B}$ for $\mathcal{H}_B$, an orthonormal basis for $\mathcal{H}_{AB}$ is

$$\{|i\rangle_A \otimes |j\rangle_B : 1 \leq i \leq d_A,\ 1 \leq j \leq d_B\},$$

often abbreviated $|i, j\rangle$ or $|ij\rangle$. Hence

$$\dim \mathcal{H}_{AB} = d_A \cdot d_B.$$

If A is prepared in state $|\psi\rangle_A$ and B in state $|\phi\rangle_B$ independently, the composite state is the **product state**

$$|\psi\rangle_A \otimes |\phi\rangle_B \in \mathcal{H}_{AB}.$$

Product states span $\mathcal{H}_{AB}$ but do not exhaust it: a generic element of $\mathcal{H}_{AB}$ is a linear combination of product states. Such non-product states are called **entangled**, and are the subject of Module 4.

Operators on subsystems extend to operators on the composite via the tensor product of operators:

$$(A \otimes B)(|\psi\rangle_A \otimes |\phi\rangle_B) := (A|\psi\rangle_A) \otimes (B|\phi\rangle_B),$$

extended by linearity to all of $\mathcal{H}_{AB}$. An operator acting on A alone is $A \otimes I_B$; one acting on B alone is $I_A \otimes B$.

Intuition

The structural point is the comparison with classical composite systems.

Structure	Classical (product of sets)	Quantum (tensor of spaces)
Compose state spaces	$S_A \times S_B$	$\mathcal{H}_A \otimes \mathcal{H}_B$
Dimension rule	$\dim(S_A \times S_B) = \dim S_A + \dim S_B$	$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim \mathcal{H}_A \cdot \dim \mathcal{H}_B$
Generic joint state	factorizable: (a, b)	generally not factorizable
Resource	simple storage	exponential superposition

Two classical bits have $2 + 2 = 4$ degrees of freedom — but they describe one of $2 \times 2 = 4$ possible states. Two quantum bits have $2 \cdot 2 = 4$ **complex amplitudes**, one for each computational basis state, all of which can be simultaneously nonzero. The *amplitude space* of n qubits has dimension 2^n, which is the structural origin of every "quantum advantage" story.

Entanglement is the statement that most vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$ are not products of anything. The tensor product enlarges the state space beyond what "independent preparation" can reach, and the extra room is what Module 4 will study.

Structural Role

Every multi-system construction in the course is built on this postulate:

- **Multi-qubit registers** (Node 7.1) live in $(\mathbb{C}^2)^{\otimes n}$.
- **Bell states and CHSH** (Node 4.4) are the simplest entangled states and violate a classical factorization bound.
- **Partial trace** (Node 5.2) reduces a joint state to a subsystem state and is the structural origin of decoherence and mixed states.
- **Environments** (Node 5.3) are modelled as tensor factors, and Module 8's open-systems dynamics arise from tracing over them.
- **Quantum channels** (Node 7.3) are constructed via Stinespring dilation: every channel is a unitary on a larger tensor-product Hilbert space followed by a partial trace.

Remove the tensor product, or replace it with a direct sum, and none of these exist. The direct sum $\mathcal{H}_A \oplus \mathcal{H}_B$ gives a space of dimension $d_A + d_B$ — which is what you would need if "composite" meant "exclusive alternative" rather than "simultaneous subsystems."

Mathematical Development

Bilinearity

The tensor product $\otimes : \mathcal{H}_A \times \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ is bilinear:

$$(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) \otimes |\phi\rangle = \alpha|\psi_1\rangle \otimes |\phi\rangle + \beta|\psi_2\rangle \otimes |\phi\rangle,$$

and similarly in the second argument.

Inner product on the tensor space

The inner product on $\mathcal{H}_A \otimes \mathcal{H}_B$ is determined by

$$\langle \psi_1 \otimes \phi_1 | \psi_2 \otimes \phi_2 \rangle_{AB} := \langle \psi_1 | \psi_2 \rangle_A \, \langle \phi_1 | \phi_2 \rangle_B,$$

extended by sesquilinearity. The product basis $\{|i\rangle \otimes |j\rangle\}$ is then orthonormal, confirming that it is a basis of a Hilbert space of dimension $d_A d_B$.

Operator tensor product

For operators A on $\mathcal{H}_A$ and B on $\mathcal{H}_B$:

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = (A|\psi\rangle) \otimes (B|\phi\rangle).$$

Key identities:

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2),$$

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger,$$

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B).$$

Local operations

An operator of the form $A \otimes I_B$ is called **local on** A: it acts on $\mathcal{H}_A$ without touching $\mathcal{H}_B$. Two local operators $A_A \otimes I_B$ and $I_A \otimes B_B$ always commute, regardless of whether A and B would commute on their own subsystem. This is the formal reason why a measurement on one half of an entangled pair does not mechanically disturb the other — the operators commute, so their order does not affect joint statistics. The subtlety of Bell-type experiments (Node 10.1) is that correlations in the outcomes nevertheless exceed what any local hidden-variable model can produce.

Entanglement vs separability — the formal test

A pure state $|\Psi\rangle_{AB}$ is **separable** (equivalently, a product state) if it can be written as $|\psi\rangle_A \otimes |\phi\rangle_B$ for some $|\psi\rangle_A, |\phi\rangle_B$. Otherwise it is **entangled**.

Criterion. Write $|\Psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle|j\rangle$ in a product basis, and form the $d_A \times d_B$ matrix C of coefficients. The state is separable if and only if $\mathrm{rank}(C) = 1$. Otherwise it is entangled, and the Schmidt decomposition (Node 4.3) gives its canonical form.

Worked Example

Two qubits

$\mathcal{H}_{AB} = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$. Basis:

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

A product state

Prepare A in $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and B in $|0\rangle$. The composite state is

$$|+\rangle_A \otimes |0\rangle_B = \tfrac{1}{\sqrt{2}}(|00\rangle + |10\rangle).$$

Coefficient matrix:

$$C = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{rank}(C) = 1.$$

Separable — as expected, since it was prepared by independent operations on the two subsystems.

A Bell state (entangled)

The Bell state

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

has coefficient matrix

$$C = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{rank}(C) = 2.$$

No factorization $|\Phi^+\rangle = |\psi\rangle_A \otimes |\phi\rangle_B$ exists. Attempting one leads to the contradiction $\psi_0\phi_0 = 1/\sqrt{2}$, $\psi_0\phi_1 = 0$, $\psi_1\phi_0 = 0$, $\psi_1\phi_1 = 1/\sqrt{2}$: the middle two force $\psi_0 = 0$ or $\phi_1 = 0$ (and similarly on the other pair), which is incompatible with the outer two being nonzero.

Local operator action

The operator $Z \otimes I$ acts on the Bell state as

$$(Z \otimes I)|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(Z|0\rangle \otimes |0\rangle + Z|1\rangle \otimes |1\rangle) = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle.$$

A local operation on A transforms the entangled state into another (still entangled) state. The structural fact here is that Bell states form an orthonormal basis of $\mathcal{H}_{AB}$, and local Pauli operators act transitively on this basis.

Cross-Links

- **Linear Algebra:** Tensor products of vector spaces, Kronecker products of matrices, rank of a matrix.
- **Quantum Information:** n-qubit register $(\mathbb{C}^2)^{\otimes n}$ and universal gate sets acting on it.
- **VQA:** Parameterized circuits act on tensor-product spaces; circuit depth and entangling structure determine expressivity.
- **Module 4 (Entanglement):** Full development of entanglement structure, measures, and monogamy.
- **Module 5 (Decoherence):** Partial trace over the environment tensor factor is the operation that turns entanglement with the environment into apparent classical mixtures.
- **Statistics:** Joint distributions on a product sample space correspond to classical (separable) composite states; entanglement has no classical analogue.

Structural Compression

Invariant: $\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim \mathcal{H}_A \cdot \dim \mathcal{H}_B$.

Mechanism: Bilinear extension over a product basis $\{|i\rangle \otimes |j\rangle\}$, with inner product and operators inherited multiplicatively.

Cross-System Echo: Cartesian product of classical state spaces gives additive dimension; tensor product of quantum state spaces gives multiplicative dimension. The exponential gap — 2^n complex amplitudes for n qubits versus $2n$ real coordinates for n classical bits — is exactly the resource that quantum computation, entanglement-based cryptography, and Bell nonlocality each exploit in different ways.

Exercises

1. Verify that the product basis $\{|i\rangle \otimes |j\rangle\}$ is orthonormal with respect to the inner product $\langle \psi_1 \otimes \phi_1 | \psi_2 \otimes \phi_2 \rangle := \langle \psi_1 | \psi_2 \rangle \langle \phi_1 | \phi_2 \rangle$.
2. Let A be a 2×2 matrix and B be a 3×3 matrix. Write out $A \otimes B$ as a 6×6 Kronecker matrix for general entries.
3. Show that $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ as operators on $\mathcal{H}_A \otimes \mathcal{H}_B$.
4. Prove that all four Bell states,

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \qquad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle),$$

are entangled, by computing the rank of each coefficient matrix.

5. Given $|\psi\rangle_{AB} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, determine whether the state is separable or entangled. If separable, write it explicitly as $|\psi\rangle_A \otimes |\phi\rangle_B$.
6. Show that $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$ using a product basis.
7. Argue, structurally, why n qubits have exponentially more states than n classical bits despite being "the same number of systems," and relate this to the multiplicative dimension rule.

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Concepts: tensor-product, hilbert-space, dimensionality, entanglement
Prerequisites: 2.1, 1.4
Enables: 4.1, 4.2, 5.2

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