# Symmetries and Conservation Laws

Quantum Foundations · Module 2 — Postulates

Node 2.6

## Position in Knowledge Graph

**System:** Quantum Foundations **Structural Domain:** Postulates of Quantum Mechanics **Node:** 2.6

Symmetries are the source of conservation laws in classical and quantum mechanics alike. In the quantum case the structural statement is sharper: a symmetry is a unitary (or anti-unitary) operator that commutes with the Hamiltonian, and its generator is an exactly conserved observable. This node closes Module 2 by connecting the dynamical postulate (2.4) to the algebraic structure of observables (2.2), setting up the appearance of symmetries in thermodynamic equilibria (Module 9) and foundational no-go theorems (Module 10).

---

## Formal Statement

A **symmetry** of a quantum system is a unitary operator $U : \mathcal{H} \to \mathcal{H}$ that commutes with the Hamiltonian:

$$[U, H] = UH - HU = 0.$$

(For time-reversal, anti-unitary operators are also permitted; Wigner's theorem guarantees that every symmetry is either unitary or anti-unitary.)

A **continuous symmetry** is a one-parameter unitary group $U(s) = e^{-iGs}$ with $G$ self-adjoint. The operator $G$ is the **generator** of the symmetry. The commutation condition $[U(s), H] = 0$ for all $s$ is equivalent to

$$[G, H] = 0.$$

**Quantum Noether theorem.** If $[G, H] = 0$, then the expectation value of $G$ is conserved under time evolution:

$$\frac{d}{dt}\langle G\rangle_{\psi(t)} = 0.$$

More strongly, the full probability distribution of $G$ (not just its mean) is conserved, because $G$ and $H$ are simultaneously diagonalizable and time evolution acts on $G$-eigenspaces by a phase.

---

## Intuition

If the Hamiltonian does not "see" a symmetry operator, then the symmetry operator does not change under time evolution. The Hamiltonian is the generator of time translation (Node 2.4); the symmetry generator $G$ is the generator of some other transformation (a spatial rotation, a phase shift, a particle exchange). When the two generators commute, the two transformations are independent, and neither flow disturbs the other.

The classical analogue is Noether's theorem: a continuous symmetry of the Lagrangian yields a conserved current. The quantum statement is not just an analogue but a sharpening: in quantum mechanics the conservation is exact (no coarse-graining), and it applies to the full operator spectrum, not just the expectation value.

Symmetries also carve Hilbert space into **irreducible representations** of the symmetry group. A state in a particular irreducible representation carries definite quantum numbers (angular momentum, parity, charge, colour), and these labels are protected by the symmetry: no Hamiltonian respecting the symmetry can mix different irreducible sectors. This is the structural origin of the periodic table, selection rules in spectroscopy, and particle classification in high-energy physics.

---

## Structural Role

Symmetries are the organizing principle of Hilbert space.

- **Invariant subspaces.** A symmetry group decomposes $\mathcal{H}$ into orthogonal invariant subspaces, each carrying an irreducible representation.
- **Good quantum numbers.** Generators that commute with $H$ are simultaneously diagonalizable with $H$; their eigenvalues label energy levels.
- **Degeneracy.** Multi-dimensional irreducible representations imply multi-dimensional eigenspaces of $H$ — all degeneracies in quantum mechanics trace either to a symmetry or to an accident.
- **Selection rules.** Transition matrix elements between states in different irreducible representations vanish for symmetry-respecting perturbations. This is the origin of "forbidden transitions."
- **Equilibrium ensembles.** In Module 9, Gibbs states take the form $e^{-\beta H}/Z$, which is a function of $H$ alone and therefore commutes with every symmetry of $H$; this is why conserved charges appear as chemical potentials.

Symmetry reasoning is a labor-saving device in computation and an ontological device in interpretation: it tells you which degrees of freedom are "really there" and which are gauge.

---

## Mathematical Development

### Minimum representation theory

A **group representation** is a homomorphism $\rho : G \to U(\mathcal{H})$: each group element $g$ is mapped to a unitary $\rho(g)$ with $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. A representation is **irreducible** if $\mathcal{H}$ has no nontrivial subspace that is invariant under all $\rho(g)$.

**Schur's lemma.** An operator that commutes with every element of an irreducible representation is a scalar multiple of the identity.

**Corollary.** If a Hamiltonian $H$ commutes with every element of a representation of $G$, then $H$ is a scalar on every irreducible subrepresentation. All states in a given irreducible block have the same energy — the symmetry enforces degeneracy.

### Conservation as exact identity

Let $[G, H] = 0$. Then for any state $|\psi\rangle$ and time $t$, using $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi\rangle$ and the commutator vanishing:

$$\langle\psi(t)|G|\psi(t)\rangle = \langle\psi|e^{iHt/\hbar}Ge^{-iHt/\hbar}|\psi\rangle = \langle\psi|G|\psi\rangle.$$

More generally, for any function $f$,

$$\langle\psi(t)|f(G)|\psi(t)\rangle = \langle\psi|f(G)|\psi\rangle,$$

so the entire probability distribution of $G$ is conserved. Contrast this with the classical case, where Noether's theorem conserves only the expectation value (or the value along a trajectory).

#### Heisenberg picture statement

In the Heisenberg picture (Node 2.4),

$$\frac{dG_H}{dt} = \frac{i}{\hbar}[H, G_H].$$

The right-hand side vanishes if and only if $[H, G] = 0$. Symmetry and conservation are therefore the same statement in two notations.

#### Anti-unitary case — time reversal

The only exception to unitarity is time reversal, which is implemented by an anti-unitary operator $T$ satisfying $THT^{-1} = H$ for a time-reversal-invariant Hamiltonian. Anti-unitarity flips the sign of imaginary parts (in particular, $T$ conjugates complex numbers), which is the algebraic reflection of time's arrow.

---

## Worked Example

#### SU(2) symmetry and angular momentum

A system of a spin-$\frac{1}{2}$ particle in a spherically symmetric potential has three conserved observables — the components of total angular momentum $\vec{J} = (J_x, J_y, J_z)$, generators of the $SU(2)$ rotation group. They satisfy

$$[J_a, J_b] = i\hbar\epsilon_{abc}J_c.$$

Spherical symmetry of the Hamiltonian means $[J_a, H] = 0$ for each component.

#### Good quantum numbers

$J^2 = J_x^2 + J_y^2 + J_z^2$ commutes with all $J_a$ (Casimir invariant) and with $H$. A complete set of commuting observables is $\{H, J^2, J_z\}$, and energy eigenstates can be labelled $|n, j, m\rangle$ with $J^2 = \hbar^2 j(j+1)$ and $J_z = \hbar m$.

#### Degeneracy

Each energy level $E_{n,j}$ is $(2j+1)$-fold degenerate — one state for each value of $m \in \{-j, -j+1, \ldots, j\}$. This is not a coincidence but an exact consequence of $SU(2)$ symmetry via Schur's lemma: the $2j+1$ states form an irreducible representation of $SU(2)$, and the Hamiltonian must be a constant on this irreducible subspace.

#### Selection rules

An electric dipole perturbation $\vec{d} \cdot \vec{E}$ is a vector operator (spin-1 tensor). The Wigner–Eckart theorem then forces matrix elements between angular-momentum states to vanish unless $\Delta j \in \{-1, 0, +1\}$ and $\Delta m \in \{-1, 0, +1\}$. Forbidden transitions in atomic spectra are exactly those that would violate this symmetry-based selection rule.

#### Qubit example

For a single qubit, the symmetry group that commutes with $H = (\hbar\omega/2)Z$ is the $U(1)$ generated by $Z$ itself. Since $[Z, Z] = 0$, the operator $Z$ is conserved. This matches Larmor precession (Node 2.4): the Bloch vector precesses around the $z$-axis, leaving the $z$-component unchanged. The full three-dimensional $SU(2)$ symmetry is broken by the field; only its $U(1)$ subgroup survives.

---

## Cross-Links

- **Linear Algebra:** Commuting operators are simultaneously diagonalizable; invariant subspaces under a set of operators.
- **Math-Intuition:** Groups, homomorphisms, and Lie algebras — the structural objects underlying symmetry.
- **Systems Thinking:** Conservation laws are system invariants; every "what stays fixed under the dynamics?" question in systems analysis has a quantum version as "what commutes with $H$?"
- **Quantum Thermodynamics (Module 9):** Conserved quantities define generalized Gibbs ensembles; chemical potentials are Lagrange multipliers for conserved charges.
- **Module 10 (Foundations):** Spontaneous symmetry breaking and the structural asymmetries of the Standard Model use this postulate as their starting point.
- **VQA:** Symmetry-preserving ansätze restrict to subspaces of the Hilbert space and give better optimization landscapes — a direct practical application of Schur's lemma.

---

## Structural Compression

**Invariant:** A unitary symmetry commutes with the Hamiltonian if and only if its generator is exactly conserved.

**Mechanism:** $[G, H] = 0$ via the Heisenberg equation $\dot{G}_H = (i/\hbar)[H, G_H]$.

**Cross-System Echo:** Classical Noether theorem conserves the value of a Noether charge along trajectories. Quantum Noether theorem conserves the entire probability distribution of the corresponding operator — a strictly stronger statement, reflecting the algebraic rather than trajectory-based nature of quantum dynamics.

---

## Exercises

1. Prove that if $[G, H] = 0$, then the expectation value $\langle\psi(t)|G|\psi(t)\rangle$ is independent of $t$.
2. Strengthen the previous result: show that if $[G, H] = 0$, then the full distribution $\{p(g_i)\}$ of outcomes of a $G$-measurement is conserved.
3. Verify the angular momentum commutation relations $[J_a, J_b] = i\hbar\epsilon_{abc}J_c$ for the spin-$\frac{1}{2}$ representation $J_a = (\hbar/2)\sigma_a$.
4. Show that $J^2$ commutes with each $J_a$. (Hint: use the commutation relations and expand.)
5. Apply Schur's lemma: if a Hermitian operator commutes with the $2 \times 2$ Pauli-generated $SU(2)$ representation on $\mathbb{C}^2$, show that it must be proportional to the identity.
6. Identify a continuous symmetry of the Hamiltonian $H = (\hbar\omega/2)Z$ on $\mathbb{C}^2$ and find its generator. How many dimensions does the symmetry group have?
7. Explain, structurally, why degeneracies in the spectrum of $H$ always signal either a symmetry of $H$ or an "accidental" coincidence, and give an example of each in two-level and three-level systems.

---

*Quantum Foundations · Module 2 — Postulates · Node 2.6*
**Concepts:** unitary-evolution, linear-operators
**Prerequisites:** 2.2, 2.4
**Enables:** 9.1, 10.6