Projective Measurements

Quantum Foundations · Module 3 — Measurement Theory

Node 3.1

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Measurement Theory **Node:** 3.1

This is the foundation node of measurement theory. The Born rule (Node 2.3) gave us *what happens when we measure an observable*; this node names the resulting object — a projective measurement — and exposes its structural anatomy. Generalized measurements (POVMs, Node 3.2) will then weaken this object in a specific, motivated way.

Formal Definition

A **projective measurement** (or **PVM**, projection-valued measure) on a Hilbert space $\mathcal{H}$ is a collection of operators

$$\{P_i\}_{i \in I}$$

with index set I (the set of possible outcomes), satisfying:

1. **Projection:** $P_i^2 = P_i$ for all i.
2. **Hermiticity:** $P_i^\dagger = P_i$ for all i.
3. **Orthogonality:** $P_i P_j = 0$ for $i \neq j$.
4. **Completeness:** $\sum_{i \in I} P_i = I$.

When the system is in state $|\psi\rangle$, the probability of outcome i is

$$p(i) = \langle \psi | P_i | \psi \rangle = \| P_i | \psi \rangle \|^2,$$

and the post-measurement state, conditional on outcome i, is

$$|\psi'\rangle = \frac{P_i |\psi\rangle}{\sqrt{p(i)}}.$$

This last equation is the **projection postulate** (also called wavefunction collapse).

A projective measurement is associated with an observable A via the spectral decomposition $A = \sum_i \lambda_i P_i$: the eigenvalues λ_i label the outcomes, and the projectors P_i onto the eigenspaces are the measurement operators.

Intuition

A projective measurement partitions Hilbert space into mutually orthogonal subspaces. The measurement asks: *which subspace is the state in?* The answer is probabilistic, weighted by how much of the state lies in each subspace, and the post-measurement state is the projection of the original state onto the answered subspace, renormalized.

The key word is **partition**. A projective measurement is a yes/no decomposition of Hilbert space into a sum of orthogonal directions. Every possible outcome carves off one direction, and these directions are mutually exclusive and collectively exhaustive.

Structural Role

Projective measurements are the textbook quantum measurement and the simplest measurement object. They are sufficient for:

- All idealized closed-system measurements.
- The Stern–Gerlach experiment and its variants.
- Computational-basis readout in quantum computing.
- The original Born-rule statement in Module 2.

But they are *not* sufficient for general measurement situations. Whenever a measurement involves an environment, an ancilla, or noise, the resulting effective measurement on the system of interest is in general not projective. This is the motivation for POVMs (Node 3.2), and Naimark's theorem (Node 3.3) will show that POVMs are exactly projective measurements on a larger system.

Mathematical Development

Probability sums to one

By completeness:

$$\sum_i p(i) = \sum_i \langle\psi|P_i|\psi\rangle = \langle\psi| \sum_i P_i|\psi\rangle = \langle\psi|I|\psi\rangle = 1.$$

Repeated measurement is consistent

Immediately re-measuring the same projective observable yields the same outcome with probability 1. Proof: after the first measurement, the state is $|\psi'\rangle = P_i|\psi\rangle/\sqrt{p(i)}$. Then

$$\langle\psi'|P_j|\psi'\rangle = \frac{\langle\psi|P_iP_jP_i|\psi\rangle}{p(i)} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

using $P_iP_j = \delta_{ij}P_i$. This is the structural reason "wavefunction collapse" is self-consistent.

Expectation values

For an observable $A = \sum_i \lambda_i P_i$, the expected outcome is

$$\langle A\rangle_\psi = \sum_i \lambda_i p(i) = \sum_i \lambda_i \langle\psi|P_i|\psi\rangle = \langle\psi|A|\psi\rangle.$$

This is the formula that VQA uses to define cost functions.

Worked Example

Computational-basis measurement of a qubit

Take the observable $Z = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$. The associated projective measurement has:

$$P_0 = |0\rangle\langle 0|, \qquad P_1 = |1\rangle\langle 1|.$$

These satisfy $P_0^2 = P_0$, $P_1^2 = P_1$, $P_0 P_1 = 0$, and $P_0 + P_1 = I$.

Measuring a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$p(0) = \langle\psi|P_0|\psi\rangle = |\alpha|^2, \qquad p(1) = |\beta|^2.$$

Conditional on outcome 0, the post-measurement state is $|0\rangle$. Conditional on outcome 1, it is $|1\rangle$. Information about the original superposition coefficients is lost — only their squared moduli were observable.

Hadamard-basis measurement of the same qubit

Take the observable $X = (+1)|+\rangle\langle +| + (-1)|-\rangle\langle -|$ where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Measuring the same $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$\langle +|\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}, \qquad p(+) = \frac{|\alpha+\beta|^2}{2}.$$

This probability depends on the **relative phase** between α and β. The same state, measured in two different bases, exposes different information — a structural feature of quantum mechanics that is responsible for things like the BB84 protocol and the entire story of complementarity.

Cross-Links

- **Linear Algebra:** Orthogonal projection onto subspaces; spectral theorem.
- **Statistics:** A projective measurement is a partition of an outcome space; the Born rule attaches probabilities. The non-commutative structure is what differs.
- **VQA:** Cost-function evaluation reduces to repeated projective measurements in the computational basis.
- **Quantum Information:** All measurement-based protocols (BB84, teleportation, etc.) use projective measurements as primitives.

Structural Compression

Invariant: A projective measurement is a complete orthogonal decomposition of identity into projectors; outcome probabilities sum to 1 by construction.

Mechanism: $\{P_i\}$ with $P_i^2 = P_i = P_i^\dagger$, $P_i P_j = 0$ for $i \neq j$, $\sum_i P_i = I$.

Cross-System Echo: Indicator functions on a partition of a sample space play exactly this role classically. The non-commutativity of projectors in different bases is what makes the quantum case structurally distinct.

Exercises

1. Verify that for $P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|$ on $\mathbb{C}^2$, all four PVM axioms hold.
2. Show that for any observable $A = \sum_i \lambda_i P_i$, the expectation value formula $\langle A \rangle = \langle \psi | A | \psi \rangle$ follows from the Born rule.
3. Construct a projective measurement on $\mathbb{C}^3$ with outcome set $\{1, 2\}$ where one projector has rank 2.
4. Show that projective measurements with the same projectors but different outcome labels are operationally equivalent up to a relabeling of the classical outcome register.
5. Argue why two projective measurements in mutually unbiased bases (e.g., Z and X on a qubit) cannot be performed simultaneously without disturbance.

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Concepts: born-rule, eigenvalue-decomposition, linear-operators
Prerequisites: 2.2, 2.3, 1.6
Enables: 3.2, 3.3

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