Naimark's Dilation Theorem

Quantum Foundations · Module 3 — Measurement Theory

Node 3.3

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Measurement Theory **Node:** 3.3

Naimark's theorem says POVMs are not really new objects. Every POVM on a system is a projective measurement on a larger system that includes an ancilla. The theorem closes the structural loop opened in Node 3.2: projective measurements remain the fundamental measurement primitive, and POVMs are their image under partial trace. It is the measurement-theoretic analogue of Stinespring dilation for channels (Node 7.3).

Formal Statement

Theorem (Naimark). Let $\{E_i\}_{i=1}^n$ be a POVM on a Hilbert space $\mathcal{H}_S$. Then there exist:

- an auxiliary Hilbert space $\mathcal{H}_A$ (ancilla),
- a fixed ancilla state $|0\rangle_A \in \mathcal{H}_A$,
- a unitary operator U on $\mathcal{H}_S \otimes \mathcal{H}_A$,
- a projective measurement $\{P_i\}_{i=1}^n$ on $\mathcal{H}_S \otimes \mathcal{H}_A$,

such that for every state $|\psi\rangle \in \mathcal{H}_S$ and every outcome i,

$$\langle\psi|E_i|\psi\rangle = \langle\psi \otimes 0_A|U^\dagger P_i U|\psi \otimes 0_A\rangle.$$

Equivalently, if $V : \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_A$ is the **isometry** $V|\psi\rangle := U(|\psi\rangle \otimes |0\rangle_A)$, then $V^\dagger V = I_S$ and

$$E_i = V^\dagger P_i V.$$

The minimal dimension of the ancilla is $\dim \mathcal{H}_A \leq n$, where n is the number of POVM outcomes.

Intuition

Every POVM is the shadow of a projective measurement cast onto a subsystem.

Operationally: you perform a POVM on a system by coupling it to an auxiliary "meter" (ancilla), turning on an interaction (unitary), and then doing a sharp projective measurement on the combined system-plus-meter. What you call the "POVM outcome on S" is really the outcome of a projective measurement on $S \otimes A$, with A silently traced over.

The fuzziness of POVMs — non-orthogonal effects, more outcomes than system dimensions — is therefore not a fundamental feature of measurement but a consequence of ignoring part of the apparatus. Add the ignored part back, and you recover a perfectly sharp projective measurement.

This is the structural statement that projective measurement is the fundamental measurement primitive of quantum mechanics. POVMs are convenient, necessary, and unavoidable in practice, but they are not independent of the basic postulates.

Structural Role

Naimark closes the loop opened by POVMs:

- **Before Naimark:** POVMs might seem to require a new measurement postulate, because projective measurements cannot produce non-orthogonal effects.
- **After Naimark:** POVMs are derived objects. The measurement postulate of the course remains projective measurement; POVMs emerge from ignoring degrees of freedom.

Downstream consequences:

- **Quantum channels (Node 7.3).** Every CPTP map is a unitary on a larger space followed by a partial trace — Stinespring dilation, the direct analogue of Naimark at the level of state dynamics.
- **Open systems (Module 8).** Markovian master equations are derived by Naimark-like reductions: couple system to environment, evolve unitarily, trace out the environment, and extract an effective dynamics.
- **Interpretations (Module 6).** Naimark is invoked by "unitary only" interpretations (Many-Worlds, consistent histories) to argue that measurement is not a separate physical process but a feature of viewing a larger unitary evolution from one subsystem's perspective.

Mathematical Development

Construction of the isometry

Given POVM $\{E_i\}_{i=1}^n$ on $\mathcal{H}_S$ with $\dim \mathcal{H}_S = d$, define the ancilla $\mathcal{H}_A = \mathbb{C}^n$ with orthonormal basis $\{|i\rangle_A\}_{i=1}^n$.

Take square roots: let $M_i := \sqrt{E_i}$, so $M_i^\dagger M_i = E_i$. Define the linear map

$$V : \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_A, \qquad V|\psi\rangle := \sum_{i=1}^n M_i|\psi\rangle \otimes |i\rangle_A.$$

V is an isometry. Compute

$$\langle V\psi|V\phi\rangle = \sum_{i,j} \langle\psi|M_i^\dagger M_j|\phi\rangle\langle i|j\rangle_A = \sum_i \langle\psi|M_i^\dagger M_i|\phi\rangle = \sum_i \langle\psi|E_i|\phi\rangle = \langle\psi|\phi\rangle,$$

using the completeness of the POVM in the last step. Therefore $V^\dagger V = I_S$.

The projective measurement on the dilated space

Define projectors $P_i := I_S \otimes |i\rangle\langle i|_A$. These satisfy $P_i^\dagger = P_i$, $P_i^2 = P_i$, $P_iP_j = 0$ for $i \neq j$, and $\sum_i P_i = I_S \otimes I_A$ — a projective measurement.

Recovery of POVM probabilities

For any $|\psi\rangle \in \mathcal{H}_S$:

$$\langle V\psi|P_i|V\psi\rangle = \langle V\psi|(I_S \otimes |i\rangle\langle i|_A)|V\psi\rangle.$$

Substituting $V|\psi\rangle = \sum_j M_j|\psi\rangle \otimes |j\rangle_A$:

$$= \sum_{j,k} \langle\psi|M_j^\dagger M_k|\psi\rangle\langle j|i\rangle_A\langle i|k\rangle_A = \langle\psi|M_i^\dagger M_i|\psi\rangle = \langle\psi|E_i|\psi\rangle.$$

Probability on the dilated space agrees with the POVM probability on the original space.

Extending V to a unitary

An isometry $V : \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_A$ with $\dim(\mathcal{H}_S \otimes \mathcal{H}_A) \geq \dim \mathcal{H}_S$ extends to a unitary U on $\mathcal{H}_S \otimes \mathcal{H}_A$ by picking an arbitrary completion of its range to an orthonormal basis of the codomain. Physically, the unitary U describes the interaction between system and ancilla that prepares the state $V|\psi\rangle$ from the factorized initial state $|\psi\rangle \otimes |0\rangle_A$.

Minimality

The ancilla dimension n (number of POVM outcomes) is the generic requirement. When POVM effects have special structure (e.g., all rank one), smaller dilations exist. The minimal dilation is unique up to unitary equivalence on the ancilla — the measurement-theoretic counterpart of the fact that Stinespring dilation is unique up to isometry on the environment.

Worked Example

The trine POVM as a projective measurement on a dilated space

Recall the qubit trine POVM (Node 3.2) with three rank-one effects:

$$E_i = \tfrac{2}{3}|\phi_i\rangle\langle\phi_i|, \quad i = 1, 2, 3.$$

The system Hilbert space is $\mathcal{H}_S = \mathbb{C}^2$; the POVM has three outcomes, so we need a three-dimensional ancilla $\mathcal{H}_A = \mathbb{C}^3$. The combined space is $\mathcal{H}_S \otimes \mathcal{H}_A \cong \mathbb{C}^6$.

Kraus operators. Set $M_i := \sqrt{2/3}\,|\phi_i\rangle\langle\phi_i|$, so $M_i^\dagger M_i = \frac{2}{3}|\phi_i\rangle\langle\phi_i| = E_i$.

Isometry.

$$V|\psi\rangle = \sum_{i=1}^{3} M_i|\psi\rangle \otimes |i\rangle_A = \sqrt{\tfrac{2}{3}} \sum_{i=1}^{3} \langle\phi_i|\psi\rangle\,|\phi_i\rangle \otimes |i\rangle_A.$$

For $|\psi\rangle = |0\rangle$, $\langle\phi_1|0\rangle = 1$, $\langle\phi_2|0\rangle = \frac{1}{2}$, $\langle\phi_3|0\rangle = \frac{1}{2}$, and the isometry places the system in a genuinely entangled state on $\mathbb{C}^6$. A projective measurement of the ancilla register then produces outcome i with probability $\langle 0|E_i|0\rangle$, matching the POVM.

A two-outcome POVM

For a two-outcome POVM with effects $E_0, E_1 = I - E_0$, a qubit ancilla suffices. The isometry is

$$V|\psi\rangle = \sqrt{E_0}|\psi\rangle \otimes |0\rangle_A + \sqrt{E_1}|\psi\rangle \otimes |1\rangle_A.$$

Measuring the ancilla in its computational basis performs the POVM on the system. This is the generic construction used in quantum tomography when a qubit ancilla is available.

Cross-Links

- **Linear Algebra:** Isometric embeddings; operator square roots; polar decomposition.
- **Quantum Information:** Stinespring dilation (Node 7.3) — the channel-level analogue of Naimark. Same structural move: "represent a generalized operation on a subsystem as a unitary on a larger system followed by a projection or trace."
- **Module 8 (Open Systems):** System-plus-environment unitary dynamics followed by partial trace generates Lindblad dynamics — a dynamical Naimark dilation.
- **Module 6 (Interpretations):** "Measurement as unitary entanglement with a meter" is the interpretive reading of Naimark used by Many-Worlds and Wigner's friend arguments.
- **Module 5 (Decoherence):** Environment-induced decoherence is Naimark dilation with the unmeasured ancilla being the environment degrees of freedom.

Structural Compression

Invariant: Every POVM on a system is a projective measurement on a larger system.

Mechanism: Isometric embedding $V : \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_A$ with $E_i = V^\dagger P_i V$, extendable to a unitary on the combined space.

Cross-System Echo: Stinespring dilation for channels, Sz.-Nagy dilation for contractions, and the general pattern "a morphism in a smaller category is a piece of a nicer morphism in a larger category." In each case, the smaller category's generalized objects are derived by restriction, not by adding a new postulate.

Exercises

1. Verify directly that the map $V : \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_A$ defined by $V|\psi\rangle = \sum_i M_i|\psi\rangle \otimes |i\rangle_A$ is an isometry when $\sum_i M_i^\dagger M_i = I_S$.
2. Show that an isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ with $\dim \mathcal{H}_2 \geq \dim \mathcal{H}_1$ can always be extended to a unitary on $\mathcal{H}_2$ (possibly after enlarging the domain to $\mathcal{H}_2$ with an auxiliary input).
3. Construct the Naimark dilation explicitly for a two-outcome qubit POVM $E_0 = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$, $E_1 = I - E_0$. Write V as a 4×2 matrix and extend it to a 4×4 unitary.
4. Show that when a POVM is already a projective measurement, the Naimark dilation is trivial: the ancilla can be one-dimensional and V is the identity on $\mathcal{H}_S$.
5. Let $\{E_i\}$ be an informationally complete POVM on $\mathbb{C}^2$ with four rank-one elements. Construct a Naimark dilation and identify the dimension of the minimal ancilla.
6. Explain why the Naimark dilation makes POVMs "consistent with" rather than "additional to" the standard measurement postulate. What are the implications for interpretations that deny the reality of wavefunction collapse?
7. State and prove the analogue of Naimark's theorem for density-matrix inputs: given a POVM $\{E_i\}$ and a state ρ, express $\mathrm{Tr}(E_i\rho)$ as the outcome probability of a projective measurement on $\rho \otimes |0\rangle\langle 0|_A$ after unitary evolution.

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Concepts: tensor-product, hilbert-space, unitary-evolution
Prerequisites: 3.2, 2.5
Enables: 8.2

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