Quantum Non-Demolition Measurement

Quantum Foundations · Module 3 — Measurement Theory

Node 3.4

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Measurement Theory **Node:** 3.4

A quantum non-demolition (QND) measurement is one whose back-action does not disturb the very quantity being measured. It is the closest quantum mechanics gets to a classical-style reading of an observable — "look and see" without perturbing what you looked at. QND measurements are the structural resource behind continuous monitoring, feedback control, and gravitational-wave detection.

Formal Definition

Let S be a system coupled to a meter M via an interaction Hamiltonian H_{int}. A measurement of an observable A_S of the system by the meter is **quantum non-demolition** (QND) with respect to A_S if:

1. **Back-action preservation.** $[A_S, H_{\text{int}}] = 0$. The interaction that transfers information about A_S to the meter does not change A_S.
2. **Self-compatibility across times.** $[A_S(t), A_S(t')] = 0$ for all measurement times t, t'. The observable at different times commutes with itself — any free evolution of the system between measurements does not mix A_S with non-commuting observables.

The second condition is automatic if $[A_S, H_S] = 0$, i.e. if A_S is a conserved quantity of the free system Hamiltonian. In that case $A_S(t) = A_S$ in the Heisenberg picture (Node 2.4), and the QND condition reduces to "the meter couples to a conserved observable."

Equivalently, in the Heisenberg picture, a QND observable satisfies

$$A_S(t) = U_{SM}^\dagger(t) A_S U_{SM}(t) = A_S,$$

where U_{SM} is the joint system–meter evolution.

Consequence. Repeated QND measurements of A_S produce the same outcome. The first measurement projects the system into an eigenstate of A_S; subsequent QND measurements do not disturb that eigenstate and therefore reproduce the outcome deterministically.

Intuition

A projective measurement always disturbs *something* — by the measurement postulate, the state is projected into an eigenspace of the measured observable. The question is *what* is disturbed. If the state was already in an eigenstate of A, the projection does nothing; otherwise, it collapses the component along the measured axis but leaves orthogonal components free to be disturbed.

A QND measurement of A is a measurement that disturbs only the conjugate observables of A, not A itself. You pay the measurement-disturbance cost in some other variable. This is the best you can do in quantum mechanics: you cannot make a measurement that disturbs nothing, but you can choose which variable absorbs the disturbance.

The price is that you must engineer your coupling to commute with the observable you care about. This is non-trivial: most natural system–meter couplings are energy-exchange interactions that disturb amplitude and phase together, and building a QND coupling is a deliberate experimental design.

Structural Role

QND measurements occupy a specific niche between projective measurement (instantaneous, strong) and weak continuous monitoring (Node 3.5):

- **Continuous monitoring** (Node 8.4, stochastic master equations): QND measurements, taken repeatedly at short intervals, produce a classical record of the QND observable's trajectory without disturbing it. This is the structural foundation of quantum trajectory theory.
- **Feedback control:** a QND measurement gives a reliable instantaneous readout that can feed back into a control loop. Non-QND measurements would perturb the control variable itself.
- **Quantum error correction:** stabilizer measurements in error correction (Node 8.6) are QND with respect to the stabilizers — this is why they can be repeated without disturbing the logical information.
- **Precision metrology:** gravitational-wave detectors and atomic clocks rely on QND measurements to push readout noise below the standard quantum limit.
- **Conservation laws $\leftrightarrow$ QND observables.** A conserved observable (one that commutes with the Hamiltonian, Node 2.6) is always a candidate for QND measurement, because its Heisenberg-picture time dependence is trivial.

Mathematical Development

Necessary and sufficient conditions

Let $H = H_S + H_M + H_{\text{int}}$ be the total Hamiltonian of system plus meter. The Heisenberg equation gives

$$\frac{dA_S}{dt} = \frac{i}{\hbar}[H, A_S] = \frac{i}{\hbar}[H_S, A_S] + \frac{i}{\hbar}[H_{\text{int}}, A_S].$$

For $A_S(t)$ to be constant under the joint evolution, both terms must vanish:

$$[H_S, A_S] = 0 \quad \text{and} \quad [H_{\text{int}}, A_S] = 0.$$

The first is the conservation condition (Node 2.6). The second is the QND coupling condition — the system–meter interaction must commute with the observable under measurement.

Back-action is redirected, not removed

The total Hamiltonian still generates unitary evolution, so back-action exists — it is simply localized away from A_S. Specifically, any observable B_S with $[B_S, A_S] \neq 0$ is in general disturbed:

$$\frac{dB_S}{dt} = \frac{i}{\hbar}[H_{\text{int}}, B_S] \neq 0 \quad \text{when } [H_{\text{int}}, A_S] = 0.$$

QND measurement transfers the "measurement cost" from A onto its non-commuting conjugates. This is an explicit version of the uncertainty principle as a design principle.

Iterated QND measurements and the projection chain

Starting from a state $|\psi\rangle$ that is not an A_S-eigenstate, the first QND measurement yields outcome λ_i with probability $\|P_i|\psi\rangle\|^2$ and leaves the system in $P_i|\psi\rangle/\|P_i|\psi\rangle\|$, an A_S-eigenstate. Every subsequent QND measurement of A_S returns λ_i with probability one and leaves the state unchanged. In this sense QND measurement "locks in" an eigenvalue, then reports it arbitrarily many times without further disturbance.

Contrast with non-QND measurement

If instead $[H_{\text{int}}, A_S] \neq 0$, the interaction shifts A_S during the measurement, and repeated measurements do *not* yield the same outcome — they sample a distribution whose width grows with measurement strength. This is the standard behaviour of energy-exchange couplings.

Worked Example

Photon-number QND via cross-Kerr coupling

A standard QND scheme for measuring the photon number $\hat{N} = a^\dagger a$ in a cavity uses a dispersive coupling to an atom (probe). The interaction Hamiltonian is

$$H_{\text{int}} = \hbar\chi\, \hat{N} \otimes \sigma_z^{\text{probe}},$$

with χ the dispersive coupling strength. Check the QND condition:

$$[H_{\text{int}}, \hat{N}] = \hbar\chi[\hat{N}, \hat{N}] \otimes \sigma_z^{\text{probe}} = 0.$$

The coupling is diagonal in the number basis of the cavity. An atom passing through the cavity picks up a phase $e^{-i\chi N\tau\sigma_z}$ proportional to the photon number, without absorbing or emitting photons. A subsequent measurement of the atomic phase reads out N; the photon number in the cavity is unchanged, and the measurement can be repeated.

LIGO: mirror position QND in principle

Gravitational-wave detectors measure the position difference of end mirrors via the phase of interfering laser beams. Naive position measurement is *not* QND: $[H_{\text{int}}, \hat{x}] \neq 0$ because the photon radiation-pressure force acts on momentum, and momentum and position are conjugate. The standard quantum limit $\Delta x \Delta p \geq \hbar/2$ follows. QND variants — "speed meters," "variational readout," "squeezed light" — redesign the interaction so that the disturbance is pushed entirely into the phase quadrature, leaving the measured quadrature uncorrupted.

Qubit Z-measurement via ancilla phase-kickback

For a qubit observable Z, a QND readout couples the qubit to an ancilla via $H_{\text{int}} = g\, Z_S \otimes X_A$. This commutes with Z_S, so the measurement is QND with respect to Z_S. The ancilla picks up a Z_S-dependent rotation; reading the ancilla then projects onto Z_S-eigenstates without disturbing that observable. This is the design behind dispersive qubit readout in circuit QED.

Cross-Links

- **Module 2.6 (Symmetries):** Conserved observables (those commuting with H_S) are automatically candidates for QND measurement. The two structural conditions align.

- **VQA:** Repeated cost-function evaluation needs the cost observable to be (approximately) QND under the measurement scheme; otherwise each shot disturbs the next one.
- **Module 8 (Open Systems):** Continuous QND measurement, stochastically averaged, generates quantum trajectories and stochastic master equations (Node 8.4).
- **Module 7 (Error Correction):** Stabilizer measurements must be QND with respect to the stabilizer group; this is the requirement that error syndromes can be extracted without disturbing the encoded logical state.
- **Systems Thinking:** "Measure without perturbing" is the canonical control-theory desideratum; QND makes it precise in the quantum case.

Structural Compression

Invariant: A measurement is QND with respect to A if and only if the system–meter interaction commutes with A.

Mechanism: $[A, H_{\text{int}}] = 0$ ensures the Heisenberg-picture value of A is preserved under the joint evolution.

Cross-System Echo: Conserved quantities in classical mechanics are unaffected by dynamical perturbations that commute (in the Poisson-bracket sense) with them; QND measurement is the same invariance condition applied to the measurement interaction, not to free evolution.

Exercises

1. Show that if $[A, H_{\text{int}}] = 0$ and $[A, H_S] = 0$, then A is conserved under the joint system–meter evolution.
2. Verify the QND condition for the photon-number coupling $H_{\text{int}} = \hbar\chi\hat{N}\otimes\sigma_z$. Show that the corresponding measurement of the cavity mode preserves $\hat{N}$ and gives outcome N with probability equal to the initial photon-number distribution.
3. Show that a position–momentum coupling $H_{\text{int}} = g\hat{x}_S\hat{p}_M$ is QND with respect to $\hat{x}_S$ but not with respect to $\hat{p}_S$. What is the back-action on $\hat{p}_S$?
4. Construct a QND scheme for measuring X on a qubit, by designing an interaction H_{int} that commutes with X_S.
5. A qubit with Hamiltonian $H_S = (\hbar\omega/2)X$ is coupled to a meter via $H_{\text{int}} = gZ_S \otimes \sigma_z^M$. Is the measurement of Z_S QND? (Hint: check both conditions.)
6. Explain why a classical measurement (pointer needle pointing at a dial) can in principle be QND with zero back-action, and why the quantum case always has back-action somewhere (even in QND measurements).
7. Argue, structurally, why stabilizer measurements in quantum error correction must be QND with respect to the stabilizer group if the scheme is to preserve the encoded logical state.

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Concepts: observability, unitary-evolution
Prerequisites: 3.1, 2.4
Enables: 3.5

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