Weak Measurement

Quantum Foundations · Module 3 — Measurement Theory

Node 3.5

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Measurement Theory **Node:** 3.5

Weak measurement extracts a small amount of information per shot in exchange for proportionally small disturbance to the state. It is a controlled trade-off between measurement strength and back-action, parameterized continuously from "no measurement at all" (identity POVM) to "full projective measurement" (strong limit). Weak measurement is the bridge between the sharp collapses of Node 3.1 and the continuous monitoring dynamics of Module 8.

Formal Definition

A **weak measurement** is a POVM whose elements are small perturbations of the identity. Concretely, a two-outcome weak measurement of an observable A has POVM elements

$$E_\pm = \tfrac{1}{2}\,(I \pm \epsilon A)$$

for a small real parameter $\epsilon \ll 1$ (and $\|\epsilon A\| < 1$ to ensure positivity). Positivity requires $|\epsilon\lambda_i| < 1$ for all eigenvalues λ_i of A; completeness $E_+ + E_- = I$ is automatic.

The Kraus operators are $M_\pm = \sqrt{E_\pm}$, and for small ϵ,

$$M_\pm \approx \tfrac{1}{\sqrt{2}}\left(I \pm \tfrac{\epsilon}{2} A + O(\epsilon^2)\right).$$

Outcome probabilities. For a state $|\psi\rangle$,

$$p_\pm = \langle\psi|E_\pm|\psi\rangle = \tfrac{1}{2}(1 \pm \epsilon\langle A\rangle_\psi).$$

The difference $p_+ - p_- = \epsilon\langle A\rangle_\psi$ carries the information about $\langle A\rangle$ extracted in one shot.

Post-measurement state. To leading order in ϵ,

$$|\psi'_\pm\rangle = \frac{M_\pm|\psi\rangle}{\sqrt{p_\pm}} = |\psi\rangle \pm \tfrac{\epsilon}{2}(A - \langle A\rangle_\psi)|\psi\rangle + O(\epsilon^2).$$

The state is perturbed by an amount proportional to ϵ, localized along the direction $(A - \langle A\rangle)|\psi\rangle$ — orthogonal to $|\psi\rangle$ at leading order.

Information–disturbance scaling

- **Information per shot.** The distinguishability of the two outcomes is $|p_+ - p_-| = O(\epsilon)$. Estimating $\langle A \rangle$ from N shots has standard deviation $\sim 1/(\epsilon\sqrt{N})$.
- **Disturbance per shot.** The change in state $\||\psi'\rangle - |\psi\rangle\| = O(\epsilon)$.

To extract the same amount of information as a strong measurement ($\epsilon = 1$), a weak measurement requires $N \sim 1/\epsilon^2$ shots. But the per-shot disturbance is $O(\epsilon)$, so aggregated disturbance over N shots is $O(\sqrt{N\epsilon^2}) = O(1)$ — the weak measurement does not suddenly become "free." What is purchased is the *distribution* of the disturbance over many shots, which is essential when one wants to monitor a state continuously without projecting it.

Intuition

A weak measurement is "barely looking." You learn a little about the observable and disturb the state a little. The amount you learn is $O(\epsilon)$; the amount you disturb is also $O(\epsilon)$. The ratio — information-gained per disturbance — is roughly constant across measurement strengths, which is the structural content of the information–disturbance tradeoff.

Three limits are worth keeping in mind:

- **$\epsilon \to 0$:** no information, no disturbance. The POVM reduces to $E_\pm = I/2$ — a trivial measurement that returns a fair coin.
- **$\epsilon = 1$ (for a ± 1 observable):** the POVM becomes projective $E_\pm = (I \pm A)/2$. This is a strong, projective measurement with full disturbance.
- **Intermediate ϵ:** a tunable fraction of the information is extracted, with a proportionally smaller disturbance.

Weak measurement therefore continuously interpolates between "no measurement" and "projective measurement," and its integrated limit over many small steps is continuous monitoring — the dynamics of Module 8's quantum trajectories.

Structural Role

Weak measurement serves three structural purposes in the course:

1. **Bridges discrete and continuous measurement.** A single strong measurement is discrete in time; a sequence of weak measurements with $\epsilon \to 0$ and shot rate $\to \infty$ approaches a continuous measurement record. This limit generates the stochastic master equations and quantum trajectories of Node 8.4.
2. **Defines the information–disturbance tradeoff.** Weak measurement is the explicit setting in which the tradeoff can be written as an identity rather than an inequality: information scales as ϵ^2, disturbance as ϵ, and the ratio is a structural invariant of the POVM.
3. **Licenses weak values (Aharonov–Albert–Vaidman).** Combined with post-selection, weak measurement gives operational meaning to the *weak value*, a complex number that can lie outside the spectrum of the observable. Weak values are interpretively contested but experimentally realized.

Weak measurement is also the workhorse of VQA cost estimation. Each shot of a VQA circuit is a single-sample estimate of an observable expectation; averaging many shots is a weak estimation of $\langle H \rangle$, and shot-noise $\sim 1/\sqrt{N}$ is the structural residue of the $O(\epsilon)$ per-shot information scaling.

Mathematical Development

Weak value (Aharonov–Albert–Vaidman)

Given an initial (pre-selected) state $|\psi_i\rangle$ and a final (post-selected) state $|\psi_f\rangle$, the **weak value** of an observable A is

$$_w\langle A\rangle := \frac{\langle\psi_f|A|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}.$$

This can be a complex number, and its real and imaginary parts can lie well outside the spectrum of A. It is what a weak-measurement-plus-post-selection experiment reads out as the average pointer displacement.

Physically: prepare $|\psi_i\rangle$, perform a weak measurement of A (small disturbance), then perform a strong projective measurement onto $|\psi_f\rangle$ and discard all shots where the final outcome is $|\psi_f^\perp\rangle$. The conditional average of the weak-measurement pointer is $\mathrm{Re}\,{}_w\langle A\rangle$. Non-conventional (complex, out-of-spectrum) weak values occur when $|\langle\psi_f|\psi_i\rangle|$ is small — post-selection amplifies the signal.

Information gain per shot

Define the Kullback–Leibler divergence between the shot's outcome distribution and a uniform reference. To leading order in ϵ,

$$D_{\mathrm{KL}}\bigl(p\,\|\,\tfrac{1}{2}\bigr) = \tfrac{1}{2}\epsilon^2\langle A\rangle_\psi^2 + O(\epsilon^4).$$

Information gain is quadratic in the weakness parameter. This is the scaling behind the $N \sim 1/\epsilon^2$ shot requirement.

State disturbance per shot

The fidelity between pre- and post-measurement states (averaged over outcomes) is

$$\mathbb{E}_i F(|\psi\rangle, |\psi_i'\rangle) = 1 - \tfrac{1}{2}\epsilon^2(\Delta A)_\psi^2 + O(\epsilon^4),$$

also quadratic in ϵ. The variance of A is the local "disturbance sensitivity": states with large ΔA are more affected by a weak A-measurement than states near A-eigenstates.

Continuous limit

Partition time into N steps of size $\Delta t = T/N$, with each step a weak measurement of strength $\epsilon = \sqrt{\gamma\,\Delta t}$. As $N \to \infty$ (hence $\epsilon \to 0$), the cumulative measurement record becomes a continuous Wiener process, and the state dynamics converge to a **stochastic Schrödinger equation**:

$$d|\psi\rangle = \bigl(-\tfrac{i}{\hbar}H\,dt - \tfrac{\gamma}{2}(A - \langle A\rangle)^2\,dt + \sqrt{\gamma}(A - \langle A\rangle)\,dW_t\bigr)\,|\psi\rangle.$$

This is the stochastic-master-equation formulation of continuous measurement (Node 8.4). The weak-measurement framework is what justifies passing from discrete POVMs to this continuous dynamical equation.

Worked Example

Weak Z-measurement on a qubit

Take a qubit in state $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$. Perform a weak Z-measurement with strength ϵ:

$$E_\pm = \tfrac{1}{2}(I \pm \epsilon Z).$$

Outcome probabilities:

$$p_+ = \tfrac{1}{2}(1 + \epsilon\cos\theta), \qquad p_- = \tfrac{1}{2}(1 - \epsilon\cos\theta).$$

The difference $p_+ - p_- = \epsilon\cos\theta = \epsilon\langle Z\rangle$, confirming that the weak measurement reports the expectation value of Z with a signal amplitude proportional to ϵ.

For $\theta = \pi/2$ (the state $|+\rangle$), $\langle Z\rangle = 0$ and the weak measurement produces a fair coin: $p_+ = p_- = 1/2$, zero signal, but the state is still slightly disturbed toward $|0\rangle$ or $|1\rangle$ depending on the outcome. Repeated weak measurements would drive the state toward a Z-eigenstate — a stochastic purification into a definite Z value.

Shot-noise estimation of $\langle Z\rangle$

Run N weak measurements on identical copies of the state and average the outcomes coded as ± 1:

$$\langle \hat{Z}\rangle = \frac{1}{N}\sum_{k=1}^{N} x_k, \qquad x_k \in \{+1, -1\}.$$

The estimator has mean $\epsilon\cos\theta$ and variance $(1 - \epsilon^2\cos^2\theta)/N \approx 1/N$ for small ϵ. Dividing by ϵ to extract $\cos\theta$, the estimation error is $\sim 1/(\epsilon\sqrt{N})$. For $\epsilon = 1$ (strong projective measurement) this is the standard $1/\sqrt{N}$ shot noise of VQA cost estimation; for smaller ϵ it is proportionally worse, as expected.

Weak value: amplified signal

Consider the pre-selection $|\psi_i\rangle = |+\rangle$ and post-selection $|\psi_f\rangle = \cos(\alpha/2)|0\rangle + \sin(\alpha/2)|1\rangle$ with α close to $\pi/2$ (so that $|\psi_f\rangle$ is nearly orthogonal to... actually nearly parallel to $|+\rangle$). For small mismatch $\delta = \pi/2 - \alpha$, compute the weak value of Z:

$${}_w\langle Z\rangle = \frac{\langle\psi_f|Z|+\rangle}{\langle\psi_f|+\rangle} = \frac{\cos(\alpha/2) - \sin(\alpha/2)}{(\cos(\alpha/2) + \sin(\alpha/2))/\sqrt{2}}.$$

As $\alpha \to \pi/2$ (nearly orthogonal post-selection would make the denominator small), this can exceed 1 in magnitude — a value outside the spectrum $\{-1, +1\}$ of Z. This amplification comes at the cost of a small post-selection success probability, and rare successful runs can yield large signal displacements.

Cross-Links

- **Statistics:** Sequential design and low-power tests — the same information-accumulation principle, with sample size scaling quadratically in desired precision.
- **VQA:** Shot-budget cost estimation is a weak-measurement protocol; shot-noise $1/\sqrt{N}$ is the structural residue of $O(\epsilon)$ per-shot signal scaling.
- **Module 8 (Open Systems):** Continuous weak measurement limit yields the stochastic master equation and quantum trajectories; Node 8.4 makes this explicit.
- **Module 6 (Interpretations):** Weak values and post-selection play a role in interpretive debates — are weak values "real" properties of the state, or statistical artifacts?

- **Module 3.6 (Back-action):** Weak measurement is the cleanest setting in which the information–disturbance tradeoff can be stated as an explicit asymptotic identity.

Structural Compression

Invariant: Information per shot $\propto \epsilon^2$; disturbance per shot $\propto \epsilon$. The two scale in lockstep, and the aggregate information is bounded by the aggregate disturbance.

Mechanism: POVM elements perturbed from the identity by an $O(\epsilon)$ multiple of the target observable; Kraus operators inherit the perturbation, giving $O(\epsilon)$ state change and $O(\epsilon^2)$ distinguishability.

Cross-System Echo: Sequential analysis in statistics accumulates $O(\sqrt{N})$ signal against $O(1)$ noise; the same scaling governs weak-measurement estimation. Gradient descent with small learning rate is an analogous "weak update" that trades per-step progress for stability — the same information-disturbance ratio structurally transposed.

Exercises

1. Verify that the weak-measurement POVM $E_\pm = \frac{1}{2}(I \pm \epsilon A)$ satisfies $E_\pm \succeq 0$ and $E_+ + E_- = I$ for $\|\epsilon A\| \leq 1$.
2. Derive the leading-order expressions for the post-measurement state and compute the fidelity $\langle\psi|\psi'_\pm\rangle$ to order ϵ^2.
3. For a qubit in state $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$, compute the outcome probabilities for a weak X-measurement, and show that the difference $p_+ - p_- = \epsilon \sin\theta$.
4. Show that N weak measurements of strength ϵ achieve the same statistical precision as $\epsilon^2 N$ strong measurements for estimating $\langle A\rangle$.
5. Compute the weak value of Z for $|\psi_i\rangle = |0\rangle$ and $|\psi_f\rangle = (|0\rangle + e^{i\varphi}|1\rangle)/\sqrt{2}$. Explain for which φ the weak value is real.
6. Derive the stochastic Schrödinger equation as the $\Delta t \to 0$ limit of a sequence of weak measurements with strength $\epsilon = \sqrt{\gamma \Delta t}$, for a single observable A and Hamiltonian H.
7. Explain, structurally, why the $1/\sqrt{N}$ shot noise of VQA cost estimation is a universal feature of Born-rule sampling, not a hardware artifact.

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Concepts: povm, expectation-value
Prerequisites: 3.1, 3.2
Enables: 3.6

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