Tensor Product States

Quantum Foundations · Module 4 — Entanglement

Node 4.1

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Entanglement **Node:** 4.1

The composite-system postulate (Node 2.5) declared that the Hilbert space of two systems is the tensor product of their individual Hilbert spaces. This node unpacks the structural consequences. The most important of them — that the tensor product contains vectors which are *not* products — is the entire reason entanglement exists.

Formal Definition

Given two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$, the **tensor product** $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the Hilbert space with orthonormal basis

$$\{|i\rangle_A \otimes |j\rangle_B\}_{i,j}.$$

A general element of $\mathcal{H}_{AB}$ is a complex linear combination

$$|\Psi\rangle_{AB} = \sum_{i,j} c_{ij}\, |i\rangle_A \otimes |j\rangle_B,$$

with coefficients $c_{ij} \in \mathbb{C}$.

A **product state** (or **separable pure state**) is a vector of the form

$$|\Psi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$$

for some $|\psi\rangle_A \in \mathcal{H}_A$ and $|\phi\rangle_B \in \mathcal{H}_B$. Equivalently, the coefficient matrix c_{ij} factorizes as $c_{ij} = \alpha_i \beta_j$, i.e., has rank 1.

A vector in $\mathcal{H}_{AB}$ that is *not* a product state is **entangled**.

Intuition

The classical analogue of a composite system is the Cartesian product: if system A can be in 2 states and system B can be in 3, the composite has $2 + 3 = 5$ parameters of state (2 for A, 3 for B). The quantum analogue is the tensor product, which has $2 \cdot 3 = 6$ basis vectors and $6 - 1 = 5$ complex parameters of state — *not* the sum but the *product* of the dimensions.

The “extra” structure that the tensor product provides — beyond what the Cartesian product would — is exactly the space of entangled states. Linear combinations of product states are not in general product states, and there is no classical analogue for this fact.

This is the structural origin of:

- The exponential dimensionality of multi-qubit Hilbert spaces.
- The computational power of quantum simulators.
- The non-local correlations of Bell states.
- The reduction of pure-state information when restricted to subsystems (Module 5).

Structural Role

This node sits between the composite-system postulate (Node 2.5) and everything in the rest of the course that involves more than one subsystem:

- **Separability vs entanglement** (4.2) is built directly on top of this distinction.
- **Schmidt decomposition** (4.3) gives a canonical form for bipartite states.
- **Bell states** (4.4) are the simplest non-trivial entangled vectors.
- **Reduced density operators** (5.2) describe what entanglement looks like from one side only.
- **Quantum information protocols** (Module 7) all operate on tensor product spaces.

Mathematical Development

Tensor product of operators

If $A : \mathcal{H}_A \to \mathcal{H}_A$ and $B : \mathcal{H}_B \to \mathcal{H}_B$ are linear operators, their tensor product $A \otimes B$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ by

$$(A \otimes B)(|\psi\rangle_A \otimes |\phi\rangle_B) = (A|\psi\rangle_A) \otimes (B|\phi\rangle_B).$$

This is extended bilinearly to general elements.

Inner product on the tensor product

$$(\langle\psi|_A \otimes \langle\phi|_B)(|\psi'\rangle_A \otimes |\phi'\rangle_B) = \langle\psi|\psi'\rangle_A \cdot \langle\phi|\phi'\rangle_B.$$

This makes $\mathcal{H}_{AB}$ a Hilbert space.

Dimensionality

$$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim \mathcal{H}_A \cdot \dim \mathcal{H}_B.$$

For n qubits, $\mathcal{H} = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{2^n}$. The exponential growth is the structural origin of “quantum advantage.”

Worked Example

Two qubits

Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$. The four basis vectors of $\mathcal{H}_{AB} = \mathbb{C}^4$ are

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle,$$

where $|ij\rangle$ is shorthand for $|i\rangle_A \otimes |j\rangle_B$.

Product state example:

$$|+\rangle_A \otimes |0\rangle_B = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle).$$

The coefficient matrix is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, which has rank 1. This factors.

Entangled state example (Bell state Φ^+):

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The coefficient matrix is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which has rank 2. It does not factor. There is no choice of single-qubit states $|\psi\rangle_A, |\phi\rangle_B$ such that $|\Phi^+\rangle = |\psi\rangle_A \otimes |\phi\rangle_B$.

This is the simplest non-trivial entangled state in physics. Every Bell test, every teleportation protocol, and every quantum-key-distribution scheme is built on it.

Cross-Links

- **Linear Algebra:** Tensor products of vector spaces; rank of a matrix.
- **Statistics:** Joint distributions $P(X, Y)$ on a product sample space; independence corresponds to product structure. Entanglement is the quantum analogue of correlation, but it is structurally stronger.
- **Quantum Information:** n-qubit registers, multi-particle states.
- **VQA:** Parameterized circuits on $(\mathbb{C}^2)^{\otimes n}$ explore exactly this exponentially large state space — and barren plateaus arise from precisely this dimensionality (Node 4.4 in VQA course).

Structural Compression

Invariant: A vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ is a product state iff its coefficient matrix has rank 1.

Mechanism: Bilinear extension of basis vector pairing.

Cross-System Echo: Joint distributions in classical statistics are products iff the variables are independent. The tensor-product analogue is sharper: rank-1 vs rank-($>$)1 of the coefficient matrix is a *structural* property, not a statistical one, and it persists regardless of which observables are eventually measured.

Exercises

1. Show that the four states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.
2. Express $|+\rangle \otimes |+\rangle$ in the computational basis. Verify it is a product state.
3. Show that the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ cannot be written as $|\psi\rangle_A \otimes |\phi\rangle_B$ for any $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$.
4. For the state $|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, determine whether it is a product state by computing the rank of its coefficient matrix.
5. Show that $\dim((\mathbb{C}^2)^{\otimes n}) = 2^n$ and explain, in one sentence, why this is the source of "quantum advantage."

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Concepts: tensor-product, hilbert-space, entanglement
Prerequisites: 2.5, 1.4
Enables: 4.2, 4.3

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