Separable vs Entangled States

Quantum Foundations · Module 4 — Entanglement

Node 4.2

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Entanglement **Node:** 4.2

Node 4.1 distinguished product states from non-product states for pure bipartite systems. For mixed states the question is subtler: a mixed state can be a *statistical mixture* of product states without itself being a product, and any such mixture should still count as classically correlated rather than entangled. The right notion is **separability**, defined by the convex hull of product states. Everything outside that convex hull is entangled, and the boundary between the two — the **separability problem** — is the central problem of finite-dimensional entanglement theory.

Formal Definition

Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a finite-dimensional bipartite Hilbert space and ρ_{AB} a density operator on it.

Pure-state case. A pure state $|\Psi\rangle_{AB}$ is **separable** (equivalently, a **product state**) if there exist $|\phi\rangle_A \in \mathcal{H}_A$ and $|\chi\rangle_B \in \mathcal{H}_B$ such that

$$|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B.$$

Otherwise it is **entangled**.

Mixed-state case. A density operator ρ_{AB} is **separable** if there exist a probability distribution $\{p_k\}$ and product states $\{\rho_A^{(k)} \otimes \rho_B^{(k)}\}$ such that

$$\rho_{AB} = \sum_k p_k \, \rho_A^{(k)} \otimes \rho_B^{(k)}.$$

Otherwise it is **entangled**. The set of separable states $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is closed and convex; entangled states are everything outside this set.

Without loss of generality, the constituent product states $\rho_A^{(k)}, \rho_B^{(k)}$ can be taken to be pure (any mixed product state can be further decomposed into pure product states). So separability is equivalent to the existence of a representation

$$\rho_{AB} = \sum_k p_k \, |\phi_k\rangle_A\langle\phi_k| \otimes |\chi_k\rangle_B\langle\chi_k|.$$

Intuition

A separable state is one Alice and Bob could prepare from scratch using only **local operations and classical communication** (LOCC). The recipe: a referee picks index k with probability p_k, tells Alice to prepare $|\phi_k\rangle$ and Bob to prepare $|\chi_k\rangle$, and walks away. Whatever ensemble of correlated states this produces is separable, by construction.

The crucial observation is that classical communication between Alice and Bob can produce *correlation* — Alice and Bob's marginal states are correlated through the shared random variable k — but not *entanglement*. Entanglement is the kind of correlation that no LOCC protocol can manufacture from product states. It must be present at the start, distributed by some non-local channel, or generated by a joint quantum interaction.

For pure states the distinction collapses: a pure bipartite state is entangled iff it is non-product, full stop. The mixed-state definition is the natural generalization that respects the convex structure of density operators: any state in the convex hull of products is "as classical as" the products that compose it.

A mixed state can look entangled in one decomposition and separable in another. The defining question is whether *any* convex decomposition into products exists, not whether the most natural one does. This makes the separability problem hard.

Structural Role

Separability is the structural threshold separating quantum-correlated states from classically-correlated ones:

- **Resource theory.** Entanglement is a resource consumed under LOCC. Separable states are "free" — they can be created at no cost — and any monotone of LOCC must be invariant on them. Node 4.5 builds entanglement measures on this skeleton.
- **Entanglement detection.** Telling whether a given ρ_{AB} is separable is, in general, NP-hard (Gurvits 2003). Practical detection uses sufficient conditions: PPT (Peres–Horodecki) for low-dimensional cases, entanglement witnesses (linear functionals separating a state from the convex hull) for the rest.
- **Bell nonlocality (Node 4.4).** Separable states cannot violate any Bell inequality. The converse is false: there exist entangled states with no nonlocal behavior (Werner's example). So entanglement is strictly weaker than Bell nonlocality, but Bell nonlocality is a sufficient certificate for entanglement.
- **Decoherence (Module 5).** A typical effect of decoherence is to drag entangled states across the separability boundary into the separable region — sometimes in finite time ("entanglement sudden death"), unlike the asymptotic decay of coherence in single-system decoherence.

Mathematical Development

Convexity and extreme points

The set $\mathcal{S}$ of separable states on $\mathcal{H}_A \otimes \mathcal{H}_B$ is the convex hull of pure product states $|\phi\rangle\langle\phi| \otimes |\chi\rangle\langle\chi|$. Its extreme points are exactly these pure product states. The full state space is the convex hull of all pure states; $\mathcal{S}$ is a strict, closed convex subset of it (strict whenever $\dim \mathcal{H}_A, \dim \mathcal{H}_B \geq 2$).

By the Carathéodory theorem in dimension $d_A^2 d_B^2 - 1$ (the real dimension of the state space), any separable state can be written as a convex combination of at most $d_A^2 d_B^2$ pure product states. This bounds the search space for explicit separable decompositions but does not make the problem tractable.

The PPT criterion (Peres–Horodecki)

Define the **partial transpose** ρ^{T_B} of ρ_{AB} by transposing only the B-indices in some chosen basis $\{|j\rangle_B\}$:

$$(\rho^{T_B})_{ij,kl} := \rho_{il,kj}.$$

The result depends on the basis, but its spectrum does not.

Theorem (Peres 1996). If ρ_{AB} is separable, then $\rho^{T_B} \succeq 0$.

Proof. If $\rho_{AB} = \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)}$, then $\rho^{T_B} = \sum_k p_k \rho_A^{(k)} \otimes (\rho_B^{(k)})^T$. Each $(\rho_B^{(k)})^T$ is a valid density operator (transposition is positive on Hermitian matrices), so ρ^{T_B} is a convex combination of valid density operators and hence positive. ■

The contrapositive is the **PPT entanglement criterion**: if ρ^{T_B} has a negative eigenvalue, ρ_{AB} is entangled.

Theorem (Horodecki 1996). For $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$, PPT is necessary *and* sufficient for separability.

In higher dimensions PPT is only necessary: there exist **PPT entangled states** ("bound entangled" states) that are entangled but cannot be detected by partial transposition. They exhibit a kind of entanglement that cannot be distilled into Bell pairs by LOCC.

Entanglement witnesses

An **entanglement witness** is a Hermitian operator W such that

$$\mathrm{Tr}(W\sigma) \geq 0 \quad \text{for all separable } \sigma, \quad \text{but} \quad \mathrm{Tr}(W\rho) < 0 \text{ for some entangled } \rho.$$

By the Hahn–Banach theorem applied to the closed convex set $\mathcal{S}$, every entangled state is detected by some witness. In practice, witnesses are how entanglement is certified in laboratory experiments — choose a Hermitian W such that $\langle W \rangle$ is measurable with available detectors and outputs a negative value precisely when the state of interest is sufficiently entangled.

LOCC monotonicity

If ρ_{AB} is separable and Λ is any LOCC operation on $\mathcal{H}_A \otimes \mathcal{H}_B$, then $\Lambda(\rho_{AB})$ is separable. LOCC cannot create entanglement from a separable state. This is the structural fact that justifies calling entanglement a *resource* and motivates all of resource-theoretic quantum information.

Worked Example

Werner states

The two-qubit **Werner state** is the one-parameter family

$$\rho_W(p) = p\,|\Phi^+\rangle\langle\Phi^+| + (1-p)\,\frac{I}{4}, \qquad p \in [0,1],$$

where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. It is the convex combination of a maximally entangled Bell state with the maximally mixed state. The parameter p is the **mixing weight** of the Bell state, also called the *Werner fidelity*.

Compute the partial transpose. In the computational basis, the matrix of $\rho_W(p)$ is

$$\rho_W(p) = \frac{1}{4}\begin{pmatrix} 1+p & 0 & 0 & 2p \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 2p & 0 & 0 & 1+p \end{pmatrix}.$$

The partial transpose with respect to B swaps the off-diagonal coherences $(0,3) \leftrightarrow (1,2)$ (in the basis ordering $00, 01, 10, 11$):

$$\rho_W(p)^{T_B} = \frac{1}{4}\begin{pmatrix} 1+p & 0 & 0 & 0 \\ 0 & 1-p & 2p & 0 \\ 0 & 2p & 1-p & 0 \\ 0 & 0 & 0 & 1+p \end{pmatrix}.$$

The block $\begin{pmatrix} 1-p & 2p \\ 2p & 1-p \end{pmatrix}$ has eigenvalues $(1-p) \pm 2p = 1+p,\ 1-3p$. The other eigenvalues are $1+p$ (twice). So the partial-transpose spectrum is $\{1+p, 1+p, 1+p, 1-3p\}/4$.

Separability boundary. The minimum eigenvalue $(1-3p)/4$ is non-negative iff $p \leq 1/3$. By the Horodecki theorem (PPT is sufficient in $\mathbb{C}^2 \otimes \mathbb{C}^2$):

- $p \leq 1/3$: $\rho_W(p)$ is separable.
- $p > 1/3$: $\rho_W(p)$ is entangled.

The threshold $p = 1/3$ is the structural "depolarizing limit" of entanglement: any Bell state mixed with more than $2/3$ units of white noise becomes classically simulable.

A separable decomposition for $p = 1/3$

At the threshold, an explicit separable decomposition is

$$\rho_W(1/3) = \frac{1}{4}\sum_{i=0}^{3} |\psi_i\rangle\langle\psi_i| \otimes |\psi_i\rangle\langle\psi_i|,$$

where $|\psi_i\rangle$ are four states forming a tetrahedron in the Bloch sphere — for instance, the four corners of an inscribed regular tetrahedron. Verification is a direct computation. This shows that the algebraic threshold from PPT is achieved by an actual LOCC-realizable preparation: pick i uniformly, tell both Alice and Bob to prepare $|\psi_i\rangle$.

A bound entangled example (gestural)

In $\mathbb{C}^3 \otimes \mathbb{C}^3$, the **Horodecki state** built from unextendible product bases gives a state with $\rho^{T_B} \succeq 0$ (PPT) that is nonetheless entangled. No LOCC protocol can extract pure Bell pairs from any number of copies of it. Bound entanglement is the structural sign that entanglement is not a single resource but a hierarchy.

Cross-Links

- **Statistics:** A separable state is the quantum analogue of a *mixture model over independent factor distributions*: a hidden categorical variable k selects a product factorization, and the marginals on A and B are coupled only through k.
- **Quantum Information:** LOCC is the operational paradigm for distributed quantum protocols; separable states are exactly the LOCC-creatable states.
- **Module 4.4 (CHSH):** Bell-inequality violation requires entanglement, but the converse fails — there are entangled states with local hidden-variable models for projective measurements.
- **Module 5 (Decoherence):** Decoherence drags states across the separability boundary, often in finite time ("entanglement sudden death").
- **Module 7 (Quantum Channels):** A channel Λ is **entanglement-breaking** iff $(\Lambda \otimes I)(\rho)$ is separable for every ρ. Such channels are exactly those with measure-and-prepare structure.

Structural Compression

Invariant: The set of separable states is the convex hull of pure product states. Entangled states are exactly the complement — what cannot be built from products by classical mixing.

Mechanism:

$$\rho_{AB} = \sum_k p_k\, \rho_A^{(k)} \otimes \rho_B^{(k)} \quad \Longleftrightarrow \quad \rho_{AB} \text{ is preparable by LOCC.}$$

Cross-System Echo: Hidden-variable models in classical statistics: a classical correlation between A and B can always be modeled by a shared latent variable k inducing conditional independence. Entanglement is precisely the correlation that *cannot* be modeled this way. Bell's theorem (Module 10) makes this structural distinction empirically testable.

Exercises

1. Verify directly that the maximally mixed two-qubit state $I/4$ is separable by exhibiting an explicit decomposition into pure product states.
2. Show that for any pure bipartite state $|\Psi\rangle$, separability (in the mixed-state sense) is equivalent to being a product state. (Hint: a pure state is an extreme point of the state space.)
3. Compute the partial transpose of the Bell state $|\Phi^+\rangle\langle\Phi^+|$ and verify that it has a negative eigenvalue. What is the value?
4. For the Werner state $\rho_W(p)$, compute $\langle\Phi^+|\rho_W(p)|\Phi^+\rangle$ and explain its relationship to the singlet fraction.
5. Construct an entanglement witness W for $|\Phi^+\rangle$ of the form $W = \alpha I - |\Phi^+\rangle\langle\Phi^+|$. Find the smallest α such that $\mathrm{Tr}(W\sigma) \geq 0$ for all separable σ. (Hint: the answer is the maximum overlap of $|\Phi^+\rangle$ with any product state, which is 1/2.)
6. Show that the partial transpose of any product state $\rho_A \otimes \rho_B$ is positive semidefinite. Conclude that the PPT condition is necessary for separability without invoking the convex-combination argument.
7. Argue, structurally, why "entangled" should not be defined as "non-product" for mixed states. What goes wrong if you adopt that definition?

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Concepts: entanglement, density-matrix, tensor-product
Prerequisites: 4.1
Enables: 4.3, 4.5

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