Entanglement Measures

Quantum Foundations · Module 4 — Entanglement

Node 4.5

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Entanglement **Node:** 4.5

How much entanglement does a state have? For pure bipartite states the answer is essentially unique up to monotone reparameterization: the **entanglement entropy** of a reduced density operator is the canonical measure. For mixed states the situation fragments — different operational tasks (creating, distilling, certifying entanglement) induce different measures, none of which agrees with any other in general. This node lays out the axiomatic structure of an entanglement measure, the canonical pure-state measure (entanglement entropy), and a survey of the major mixed-state generalizations.

Formal Definition

Axioms of an entanglement measure

An **entanglement measure** is a function $E : \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathbb{R}_{\geq 0}$ satisfying at least:

1. **Vanishing on separable states.** $E(\rho) = 0$ for every separable ρ.
2. **LOCC monotonicity.** $E(\Lambda(\rho)) \leq E(\rho)$ for every LOCC operation Λ. (Often strengthened to *strong monotonicity*: average non-increase under stochastic LOCC.)

Common additional desiderata:

3. **Convexity.** $E(\sum_k p_k \rho_k) \leq \sum_k p_k E(\rho_k)$. Mixing should not increase entanglement.
4. **Continuity.** E is continuous in ρ (in trace norm).
5. **Asymptotic additivity.** $E(\rho^{\otimes n}) = nE(\rho)$ — consistent treatment of n independent copies.
6. **Normalization.** $E(|\Phi^+\rangle\langle\Phi^+|) = \log 2 = 1$ bit for the canonical Bell state.

A function satisfying (1) and (2) is often called an *entanglement monotone*; the stronger desiderata pick out specific physically meaningful measures.

Pure-state entanglement entropy

For a pure bipartite state $|\Psi\rangle_{AB}$ with reduced state $\rho_A := \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|$, the **entanglement entropy** is

$$E(|\Psi\rangle) := S(\rho_A) = -\,\mathrm{Tr}(\rho_A \log_2 \rho_A) = -\sum_i \lambda_i^2 \log_2 \lambda_i^2,$$

where $\{\lambda_i^2\}$ are the squared Schmidt coefficients of $|\Psi\rangle$ (Node 4.3). By the symmetry of the Schmidt decomposition, $S(\rho_A) = S(\rho_B)$, so the measure is symmetric in $A \leftrightarrow B$.

For pure states this is the **unique** asymptotically continuous, normalized, additive entanglement measure: any other reasonable measure equals it (for pure bipartite states) up to choice of logarithm base.

Mixed-state generalizations

For mixed bipartite states the entropy of the reduced state is *not* a valid entanglement measure — a separable but correlated state has nonzero $S(\rho_A)$ too. The standard mixed-state measures are:

- **Entanglement of formation:** $E_F(\rho) := \min \sum_k p_k S(\rho_A^{(k)})$, minimized over decompositions $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ into pure states. This is a *convex roof* construction.
- **Distillable entanglement:** $E_D(\rho) :=$ the asymptotic rate at which Bell pairs can be extracted from many copies of ρ by LOCC.
- **Relative entropy of entanglement:** $E_R(\rho) := \min_{\sigma \in \mathcal{S}} S(\rho \| \sigma)$, the quantum relative entropy from ρ to the closest separable state.
- **Logarithmic negativity:** $E_N(\rho) := \log_2 \|\rho^{T_B}\|_1$, built from the partial-transpose negativity.

These coincide on pure states ($= S(\rho_A)$) but generically disagree on mixed states.

Intuition

Entanglement is a *resource*, and resource theories ask "how much of it do you have?". The answer depends on what you want to *do* with it. Two natural operational meanings:

- **Cost.** How many Bell pairs must I consume to make a copy of this state by LOCC? → **entanglement of formation**.
- **Yield.** How many Bell pairs can I extract from this state by LOCC? → **distillable entanglement**.

For pure states these two converge to the same number — the entanglement entropy. For mixed states they generically don't, and there are even **bound entangled** states for which the cost is positive but the yield is zero (Node 4.2).

The pure-state case is the "thermodynamic limit" of entanglement theory: in the asymptotic regime of many independent copies, all reasonable entanglement measures collapse to one number, the entropy of the reduced state. The mixed-state case is "non-equilibrium" entanglement theory, where conversion between forms is generally lossy and the resource splinters into a hierarchy.

The structural picture is that the entanglement entropy plays the same role for entanglement that thermodynamic entropy plays for energy: it counts the *exchangeable units* of the resource in the asymptotic regime.

Structural Role

Entanglement measures provide the resource-theoretic backbone of quantum information:

- **Quantification.** They turn the qualitative question "is this state entangled?" into a quantitative one with operational meaning.
- **Conversion.** They bound how much entanglement can be transformed from one form to another by LOCC. The "second law of entanglement" ($E_F \geq E_D$ always) parallels the second law of thermodynamics.
- **Quantum-classical boundary.** Entanglement entropy of subsystems is the order parameter that diagnoses topological and conformal phases of matter (area-law vs volume-law entanglement).
- **Holography (AdS/CFT).** The Ryu–Takayanagi formula relates the entanglement entropy of a boundary region to the area of a minimal surface in the bulk geometry. This is one of the most striking structural appearances of entanglement entropy outside quantum information theory.
- **Decoherence (Module 5).** Open-system dynamics typically transform high-entanglement-entropy pure states into separable mixed states, with the entropy "leaking" into system-environment correlations.
- **VQA.** Trainability of variational quantum circuits depends sensitively on entanglement growth — barren plateaus arise when the circuit's outputs become almost-maximally entangled with auxiliary degrees of freedom.

Mathematical Development

Why the Schmidt spectrum determines pure-state entanglement

Two pure bipartite states are convertible into each other by local unitaries $U_A \otimes U_B$ iff they have the same Schmidt coefficients (Node 4.3). Any entanglement measure invariant under local unitaries (a basic requirement) is therefore a function of the Schmidt spectrum alone. Adding asymptotic additivity and continuity narrows it down to the entropy.

Theorem (Donald–Horodecki–Rudolph 2002, refining Bennett–DiVincenzo–Smolin–Wootters 1996). Among entanglement measures that are normalized, asymptotically continuous, and additive on tensor products of pure bipartite states, the entanglement entropy is unique.

LOCC monotonicity of entropy on pure states

If $|\Psi'\rangle$ is obtained from $|\Psi\rangle$ by LOCC with probability p (for pure-to-pure conversions, this means deterministic conversion via local unitaries plus probabilistic mixing), then by **Nielsen's theorem** (1999),

$$|\Psi\rangle \xrightarrow{\text{LOCC}} |\Psi'\rangle \qquad \Longleftrightarrow \qquad \vec{\lambda}(\Psi) \prec \vec{\lambda}(\Psi'),$$

where $\vec{\lambda}(\Psi) = (\lambda_1^2, \lambda_2^2, \ldots)$ is the squared Schmidt vector and $\prec$ denotes **majorization** ($\vec{\lambda} \prec \vec{\mu}$ iff every cumulative sum of $\vec{\lambda}$ is bounded by the corresponding cumulative sum of $\vec{\mu}$).

The entanglement entropy is Schur-concave (decreases under majorization), so deterministic LOCC $|\Psi\rangle \to |\Psi'\rangle$ requires $S(\rho_A) \geq S(\rho'_A)$. This is the LOCC-monotonicity statement for entropy.

Concentration and dilution

- **Concentration.** n copies of a partially entangled pure state $|\Psi\rangle$ can be converted, asymptotically, into approximately $nS(\rho_A)$ Bell pairs by LOCC. This is *entanglement concentration* (Bennett–Bernstein–Popescu–Schumacher).
- **Dilution.** Conversely, m Bell pairs can be converted into approximately $m/S(\rho_A)$ copies of $|\Psi\rangle$ by LOCC. This is *entanglement dilution*.

The two rates match: pure-state entanglement is asymptotically reversible, and the conversion rate is exactly $S(\rho_A)$.

Entanglement of formation and the convex roof

For a mixed state ρ, define

$$E_F(\rho) := \inf_{\{p_k, |\psi_k\rangle\}} \sum_k p_k S(\mathrm{Tr}_B\, |\psi_k\rangle\langle\psi_k|)\,,$$

the infimum over all pure-state ensembles $\{p_k, |\psi_k\rangle\}$ realizing $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. The **convex roof** structure ensures vanishing on separable states (which have decompositions into product states with zero Schmidt entropy) and convexity by construction.

For two qubits, Wootters (1998) gave a closed-form expression in terms of the **concurrence** $C(\rho) \in [0,1]$:

$$E_F(\rho) = h\left(\tfrac{1+\sqrt{1-C(\rho)^2}}{2}\right), \qquad h(x) := -x\log_2 x - (1-x)\log_2(1-x),$$

with $C(\rho) := \max\{0, \sqrt{\mu_1} - \sqrt{\mu_2} - \sqrt{\mu_3} - \sqrt{\mu_4}\}$ and μ_i the eigenvalues (in decreasing order) of $\rho\tilde{\rho}$, where $\tilde{\rho} := (Y \otimes Y)\rho^*(Y \otimes Y)$. This gives the unique closed-form mixed-state entanglement measure for two qubits.

Distillable entanglement and bound entanglement

$E_D(\rho)$ is defined as the supremum, over all LOCC protocols, of the asymptotic rate m/n at which n copies of ρ can be converted into m Bell pairs with vanishing error. It satisfies $E_D(\rho) \le E_F(\rho)$, with strict inequality for many mixed states. For PPT-entangled states (Node 4.2) the distillable entanglement is exactly zero, even though entanglement of formation is positive — these are the **bound entangled** states.

Logarithmic negativity

The simplest computable mixed-state monotone is the **logarithmic negativity**:

$$E_N(\rho) := \log_2 \|\rho^{T_B}\|_1,$$

where $\|\cdot\|_1$ is the trace norm. This is positive iff the partial transpose has negative eigenvalues, vanishing on PPT states (which include all separable states). It is an entanglement monotone but is *not* convex, and does not coincide with the entropy on pure states. Its main virtue is computability: no minimization or convex roof is required.

Worked Example

Bell state entropy

For $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the reduced state is

$$\rho_A = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1| = \tfrac{I}{2}.$$

Entanglement entropy: $S(\rho_A) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1$ bit.

This is the maximum possible entanglement entropy for a two-qubit state, since ρ_A is maximally mixed in dimension 2 and $\log_2 2 = 1$. All four Bell states share this entropy.

Partially entangled pure state

For $|\Psi\rangle = \cos(\theta/2)|00\rangle + \sin(\theta/2)|11\rangle$, the reduced state is

$$\rho_A = \cos^2(\theta/2)|0\rangle\langle 0| + \sin^2(\theta/2)|1\rangle\langle 1|,$$

with entropy $h(\cos^2(\theta/2))$, the binary entropy. At $\theta = \pi/2$ this equals 1 (Bell state); at $\theta = 0$ it equals 0 (product state $|00\rangle$). The function interpolates monotonically.

Werner state entanglement of formation

For the Werner state $\rho_W(p) = p|\Phi^+\rangle\langle\Phi^+| + (1-p)I/4$ (Node 4.2), Wootters' formula gives

$$C(\rho_W(p)) = \max\{0, (3p-1)/2\}.$$

So the entanglement of formation is

$$E_F(\rho_W(p)) = \begin{cases} 0 & p \le 1/3 \\ h\left(\frac{1+\sqrt{1-((3p-1)/2)^2}}{2}\right) & p > 1/3, \end{cases}$$

continuous in p, zero at the separability boundary, monotonically increasing toward 1 bit at $p = 1$. The structural threshold $p = 1/3$ coincides with the PPT/separability threshold derived in Node 4.2.

Bell state vs. classical correlation

Compare $|\Phi^+\rangle\langle\Phi^+|$ (Bell state, 1 bit of entanglement entropy) with $\frac{1}{2}(|00\rangle\langle 00|+|11\rangle\langle 11|)$ (classically correlated, separable). Both have the same reduced state $\rho_A = I/2$ and so the same *von Neumann entropy of the reduction*. But the second is separable: it has zero entanglement of formation. The entropy of the reduced state is the same, but its *significance* differs. This is why $S(\rho_A)$ is a valid entanglement measure only for *pure* bipartite states.

Cross-Links

- **Statistics:** Mutual information $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ generalizes Shannon mutual information; for pure bipartite states it equals $2S(\rho_A)$.
- **Information Theory:** Shannon entropy is the classical analogue of von Neumann entropy; entanglement entropy is the part of quantum mutual information that has no classical realization.
- **Module 4.3 (Schmidt decomposition):** Entanglement entropy is a function of the Schmidt spectrum.
- **Module 4.6 (Monogamy):** Entanglement measures (especially squared concurrence) satisfy monogamy inequalities like CKW.
- **Module 5 (Decoherence):** Decoherence typically reduces entanglement of formation to zero, often in finite time.
- **Module 7 (Quantum Information):** Channel capacities (entanglement-assisted, quantum, private) are formulated as regularized entropy quantities; entanglement measures are the resource currencies.

Structural Compression

Invariant: LOCC monotonicity: any reasonable entanglement measure is non-increasing under local operations and classical communication. Vanishing on separable states is the boundary condition.

Mechanism: For pure states, entanglement entropy $S(\rho_A) = -\sum_i \lambda_i^2 \log \lambda_i^2$. For mixed states, convex-roof extensions (entanglement of formation), regularized rates (distillable entanglement), or norm-based monotones (negativity).

Cross-System Echo: Statistical entropy quantifies *uncertainty* about a system; entanglement entropy quantifies *correlation* with an unobserved partner. The same mathematical object — von Neumann entropy of a reduced density operator — plays both roles, depending on whether the larger system is regarded as a closed pure state or as a mixed ensemble.

Exercises

1. Compute the entanglement entropy of $|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3$.
2. Show that the entanglement entropy is invariant under local unitaries $U_A \otimes U_B$.
3. For the partially entangled state $|\Psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, plot $S(\rho_A)$ as a function of θ. Where is the maximum?
4. Show that for any pure state $|\Psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, $S(\rho_A) \leq \log_2 d$, with equality iff $|\Psi\rangle$ is maximally entangled.
5. Verify that the classically correlated state $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ is separable but has $S(\rho_A) = 1$ bit. Why doesn't this count as 1 bit of entanglement?
6. Compute the logarithmic negativity $E_N(\rho_W(p))$ of the Werner state and compare its dependence on p with the entanglement of formation.
7. Argue, structurally, why pure-state entanglement is asymptotically reversible (one number suffices) while mixed-state entanglement is not (irreducible hierarchy of measures).

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Concepts: von-neumann-entropy, entanglement, density-matrix
Prerequisites: 4.2, 4.3, 5.2
Enables: 4.6

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