Monogamy of Entanglement

Quantum Foundations · Module 4 — Entanglement

Node 4.6

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Entanglement **Node:** 4.6

Quantum entanglement cannot be freely shared. If two parties A and B are maximally entangled, then A cannot be entangled with any third party C at all. This is structurally different from classical correlation, which can be shared arbitrarily: three people can be perfectly correlated on a shared random bit. Monogamy is one of the sharpest structural distinctions between quantum and classical information, and it is the structural reason quantum key distribution is secure — any eavesdropper's entanglement with the transmitted state costs entanglement between the legitimate parties in a quantifiable way.

Formal Statement

The CKW (Coffman–Kundu–Wootters) inequality

Let A, B, C be three qubits in a joint state ρ_{ABC}. Define the **tangle** $\tau(\rho) := C(\rho)^2$, the squared concurrence (Node 4.5). Then

$$\tau(A, B) + \tau(A, C) \leq \tau(A, BC),$$

where $\tau(A, B)$ and $\tau(A, C)$ are the tangles of the two-qubit reduced states $\rho_{AB} = \mathrm{Tr}_C\, \rho_{ABC}$ and $\rho_{AC} = \mathrm{Tr}_B\, \rho_{ABC}$, and $\tau(A, BC)$ is the tangle of the bipartition A versus (B, C) treated as a single four-dimensional subsystem.

The inequality is **tight**: it is saturated by the GHZ state, and its deficit (the difference between right and left sides) is called the **three-tangle** or **residual tangle**,

$$\tau_{ABC} := \tau(A, BC) - \tau(A, B) - \tau(A, C),$$

which is a genuine measure of three-party entanglement that cannot be captured by bipartite tangles alone.

General formulation

For higher dimensions and larger numbers of parties, the CKW inequality has been generalized in various ways. The cleanest version is due to Osborne and Verstraete (2006):

$$\sum_{i=2}^{n} \tau(A_1, A_i) \leq \tau(A_1, A_2 \cdots A_n)$$

for any n-qubit pure state. The structural content is the same: the total bipartite entanglement between one party and each of the others is bounded above by the bipartite entanglement between that party and the rest of the system considered as a whole.

Not every entanglement measure is monogamous in this sense; for instance, entanglement of formation is *not* monogamous on three qubits. Squared concurrence (the tangle) is the canonical choice for which the CKW inequality holds.

Intuition

Classical correlations are sharable without limit. If Alice, Bob, and Charlie each hold a copy of a shared random bit x, Alice is perfectly correlated with Bob *and* perfectly correlated with Charlie, simultaneously. There is no upper bound on how many parties can share the same classical information.

Quantum entanglement cannot be copied this way, structurally. The reason traces back to the **no-cloning theorem** (Node 7.2): if Alice could share a maximally entangled Bell pair with both Bob and Charlie simultaneously, she could (in effect) clone the Bell pair by tracing out one side, violating no-cloning. Monogamy is the quantitative refinement of this impossibility: not just "you can't share Bell pairs perfectly with two partners," but "the more entanglement you have with Bob, the less you can have with Charlie," with an exact trade-off relation.

The paradigmatic fact: **if A is in a pure entangled state with B, then C is completely uncorrelated with A**. Mathematically, if ρ_{AB} is pure, then $\rho_{ABC} = \rho_{AB} \otimes \rho_C$, so C factorizes off. There is no room for A to be entangled with anything else when its entanglement with B is already "saturated."

The CKW inequality makes this precise for partial entanglement: whatever entanglement A has with B subtracts from the entanglement it can have with C, in a specific quadratic way.

Structural Role

Monogamy is one of the load-bearing structural facts of quantum information theory:

- **Quantum key distribution (QKD) security.** In BB84 or E91 protocols, Alice and Bob measure correlated halves of a Bell state. If an eavesdropper Eve has entanglement with the qubits in transit, monogamy forces a reduction in Alice–Bob correlations that Alice and Bob can detect via Bell tests. The security of QKD is literally the CKW inequality applied to Alice–Bob–Eve.
- **No-cloning reinforcement.** Monogamy is the continuous-valued form of no-cloning. No-cloning says you can't copy arbitrary quantum states; monogamy says that even partial copies (approximate cloners) must degrade the original's entanglement with the environment.
- **Constraint on multi-party entanglement.** Monogamy means that entanglement in an n-party system is a constrained resource; a party cannot be simultaneously maximally entangled with all others. This disciplines the structure of multipartite entanglement into hierarchies like GHZ-class vs. W-class.
- **Holography and the ER=EPR conjecture.** In AdS/CFT, the bulk geometry dual to an entangled boundary state has a wormhole structure. Monogamy of entanglement on the boundary corresponds to the topological fact that one side of a wormhole can connect to only one other side — a striking structural alignment between quantum information and spacetime geometry.
- **Many-body physics.** Monogamy constrains the entanglement structure of ground states of local Hamiltonians and is behind the emergence of area laws for entanglement entropy in gapped systems.

Mathematical Development

Proof sketch of CKW for three qubits

Coffman, Kundu, and Wootters (2000) proved the inequality for pure three-qubit states by direct calculation using Wootters' closed-form expression for the concurrence. The key steps:

1. For a pure three-qubit state $|\psi\rangle_{ABC}$, the bipartition $A|BC$ treats BC as a four-dimensional system and $\tau(A, BC) = 4 \det \rho_A$, where $\rho_A = \mathrm{Tr}_{BC} |\psi\rangle\langle\psi|$. For a qubit, $4 \det \rho_A = 1 - |\vec{r}|^2$ where $\vec{r}$ is the Bloch vector.
2. The two-qubit reduced states ρ_{AB}, ρ_{AC} are generally mixed; their concurrences are computed by Wootters' formula.
3. Algebraic manipulation (lengthy but direct) shows $\tau(A, B) + \tau(A, C) \leq 4 \det \rho_A = \tau(A, BC)$, with equality iff the three-tangle vanishes.

The extension to mixed states is by convex roof: the CKW inequality holds for any convex roof extension of the squared concurrence, and equals the pure-state expression on extremal points.

Three-tangle as a genuine tripartite invariant

The residual $\tau_{ABC} := \tau(A, BC) - \tau(A, B) - \tau(A, C)$ measures the entanglement that is *not* captured by any of the bipartite reductions. It is symmetric under permutation of A, B, C (a nontrivial fact), is invariant under local unitaries, and vanishes on states that are in the "W class" (discussed below). It characterizes a genuinely multipartite form of entanglement.

Why the squared concurrence?

Not every entanglement monotone satisfies a monogamy inequality. Entanglement of formation E_F, despite being convex and monotonic under LOCC, is **not monogamous**: there exist three-qubit states with $E_F(A, B) + E_F(A, C) > E_F(A, BC)$. The tangle (squared concurrence) is special because its convex-roof structure interacts correctly with the bipartition. Later generalizations (Bai et al., 2014) showed that *any* α-th power of the concurrence with $\alpha \geq 2$ is monogamous, while $\alpha < 2$ fails.

Structurally, monogamy is a feature of the *geometry* of state space under specific measures, not a universal property of "entanglement." The right way to read the CKW inequality is as picking out τ as the *monogamous* measure, rather than claiming that entanglement in general is monogamous.

The Osborne–Verstraete extension

For pure states of n qubits, Osborne and Verstraete proved:

$$\sum_{i=2}^{n} \tau(A_1, A_i) \leq \tau(A_1, A_2 \cdots A_n) = 4 \det \rho_{A_1}.$$

The right side is bounded by 1 (for a qubit), so the total tangle of A_1 distributed among the remaining $n - 1$ parties is bounded by 1. This is the quantitative form of "you cannot simultaneously maximally entangle one qubit with many others."

Worked Example

GHZ state

The three-qubit GHZ state

$$|\mathrm{GHZ}\rangle = \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

has reduced states

$$\rho_{AB} = \tfrac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \qquad \rho_A = \tfrac{1}{2} I.$$

The two-qubit reduced state ρ_{AB} is separable (a classical mixture of $|00\rangle$ and $|11\rangle$), so its concurrence $C(A,B) = 0$ and tangle $\tau(A,B) = 0$. By symmetry $\tau(A,C) = 0$. The bipartite tangle is

$$\tau(A, BC) = 4 \det \rho_A = 4 \cdot \tfrac{1}{4} = 1.$$

CKW inequality:

$$\tau(A,B) + \tau(A,C) = 0 \leq 1 = \tau(A, BC),$$

satisfied with maximum deficit. The three-tangle is $\tau_{ABC} = 1$ — the GHZ state has all of its entanglement in the genuinely tripartite mode.

The structural picture: GHZ distributes entanglement so that *no pair* of qubits is entangled when the third is traced out, yet the global state is maximally entangled against any single-qubit cut. This is genuinely "non-bipartite" entanglement, undetectable by any bipartite measure.

W state

The three-qubit W state

$$|W\rangle = \tfrac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

has reduced states

$$\rho_{AB} = \tfrac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho_A = \tfrac{1}{3}|1\rangle\langle 1| + \tfrac{2}{3}|0\rangle\langle 0|.$$

Wootters' formula gives $C(A,B) = 2/3$, so $\tau(A,B) = 4/9$. By symmetry $\tau(A,C) = 4/9$. Compute

$$\tau(A, BC) = 4 \det \rho_A = 4 \cdot \tfrac{2}{3} \cdot \tfrac{1}{3} = \tfrac{8}{9}.$$

CKW inequality:

$$\tau(A,B) + \tau(A,C) = \tfrac{4}{9} + \tfrac{4}{9} = \tfrac{8}{9} = \tau(A, BC).$$

The inequality is **saturated**, and the three-tangle of the W state is *zero*:

$$\tau_{ABC}(W) = 0.$$

Structurally: the W state distributes its entanglement entirely into bipartite correlations — every pair is entangled, but there is no genuinely tripartite residue. This is the **W class** of tripartite entanglement, distinct from the GHZ class: local unitaries cannot convert between the two, even with finite probability.

Taxonomy of three-qubit entanglement. Tripartite pure states come in two structurally different families:

- **GHZ class:** $\tau_{ABC} > 0$. Genuinely tripartite entanglement. Tracing out any party destroys all remaining correlation.
- **W class:** $\tau_{ABC} = 0$. Pairwise entanglement only. Robust to the loss of one party — the remaining pair is still entangled.

This is the structural explanation for why "a Bell pair plus a spectator" (product class), the W state (pairwise), and the GHZ state (tripartite) are the irreducible patterns of three-qubit entanglement.

QKD security sketch

In an E91-style protocol, Alice and Bob share Bell pairs distributed through a channel. An eavesdropper Eve can, in principle, intercept and entangle an ancilla with each pair. The tripartite state is $|\Psi\rangle_{ABE}$, and monogamy bounds:

$$\tau(A,B) + \tau(A,E) \leq \tau(A,BE).$$

If Alice and Bob verify by Bell tests that their two-party tangle is close to 1 (maximal entanglement), then $\tau(A,E)$ is forced close to 0 — Eve cannot be entangled with A. No entanglement with A means Eve cannot extract information about the shared key in a coherent way. This is the **structural origin of QKD security**.

Cross-Links

- **Module 4.5 (Entanglement measures):** The squared concurrence (tangle) is the monogamous measure; not all entanglement measures share this property.
- **Module 7.2 (No-cloning):** Monogamy is the continuous quantitative form of the no-cloning theorem.
- **Quantum Information:** QKD security proofs rely on monogamy to bound an eavesdropper's correlations.
- **Holography:** ER=EPR and Ryu–Takayanagi align monogamy with the topological structure of bulk wormholes.
- **Statistics:** Classical mutual information is not monogamous — perfect correlation can be shared freely. Monogamy is a structural feature with no classical analogue.

Structural Compression

Invariant:

$$\tau(A,B) + \tau(A,C) \leq \tau(A,BC).$$

Bipartite entanglements sharing a common party satisfy a quadratic constraint.

Mechanism: For three-qubit pure states, direct algebraic proof using Wootters' concurrence formula. Generalizations via convex-roof extensions and the Osborne–Verstraete theorem for n qubits.

Cross-System Echo: No classical analogue. Classical correlations and mutual information are freely sharable; entanglement is not. Monogamy is one of the structural features of quantum correlation that most sharply distinguishes it from classical statistics, and is the quantitative backbone of quantum cryptography.

Exercises

1. Verify by direct computation that the two-qubit reduced state ρ_{AB} of the GHZ state is separable, and hence has zero concurrence.
2. Compute the reduced states $\rho_{AB}, \rho_{AC}, \rho_A$ of the W state and verify the concurrence values $C(A,B) = C(A,C) = 2/3$.
3. Show that for any pure tripartite state, $\tau(A,BC) = 4 \det \rho_A$ where ρ_A is the single-qubit reduced state. (Hint: the bipartition is pure, so use Schmidt decomposition and compute directly.)
4. Construct a three-qubit state that saturates the CKW inequality with $\tau(A,B) = \tau(A,C) = \tau(A,BC)/2$. Is it in the GHZ class or the W class?
5. Show that entanglement of formation is *not* monogamous by giving a three-qubit example with $E_F(A,B) + E_F(A,C) > E_F(A,BC)$. (This requires looking up a counterexample from the literature.)

6. Explain, using monogamy, why an eavesdropper Eve cannot perfectly clone a qubit Alice sends to Bob without destroying the Alice–Bob entanglement needed to detect her. Make the bound quantitative.
7. Argue, structurally, why monogamy of entanglement is a direct consequence of the no-cloning theorem, and why its *quantitative* form (CKW) goes beyond no-cloning by bounding partial correlations rather than just exact cloning.

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Concepts: entanglement, mutual-information
Prerequisites: 4.5

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