Reduced Density Operator and Partial Trace

Quantum Foundations · Module 5 — Decoherence

Node 5.2

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Decoherence **Node:** 5.2

The reduced density operator is the answer to the question: *given a state ρ_{AB} on a composite system, what is the state of subsystem A alone?* The answer is non-trivial precisely because entanglement exists — tracing out B from an entangled pure state does not leave a pure state on A. The partial trace is the mathematical operation that implements this restriction, and it is the bridge between the closed-system unitarity of Module 2 and the open-system dynamics of Modules 5–8.

Formal Definition

Let ρ_{AB} be a density operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. The **reduced density operator** on A, also called the marginal state on A, is defined by the **partial trace** over B:

$$\rho_A := \mathrm{Tr}_B(\rho_{AB}) := \sum_j (I_A \otimes \langle j_B|)\, \rho_{AB}\, (I_A \otimes |j_B\rangle),$$

where $\{|j_B\rangle\}$ is any orthonormal basis of $\mathcal{H}_B$. The result is independent of the chosen basis and is a valid density operator on $\mathcal{H}_A$ (positive, Hermitian, unit trace).

Equivalently, on product operators the partial trace acts as

$$\mathrm{Tr}_B(A_S \otimes B_S) = A_S\, \mathrm{Tr}(B_S),$$

and extends by linearity to arbitrary operators on $\mathcal{H}_A \otimes \mathcal{H}_B$.

The defining operational property

The reduced density operator is the *unique* operator on $\mathcal{H}_A$ such that, for every observable M_A acting on A alone,

$$\mathrm{Tr}(\rho_A M_A) = \mathrm{Tr}\big(\rho_{AB}(M_A \otimes I_B)\big).$$

Expressed differently: ρ_A reproduces the expectation values of all A-only observables computed on the full joint state. This is the sense in which ρ_A is “the state of A”: it is everything an observer confined to A needs to compute predictions.

Intuition

If Alice holds subsystem A and Bob holds subsystem B, and Alice has no access to B at all, then every measurement Alice can perform is of the form $M_A \otimes I_B$. The density operator ρ_A is the state that reproduces the statistics of all such measurements — it is Alice's complete local description of reality.

The operational meaning of the partial trace is *marginalization*: discarding subsystem B and asking what remains. In classical probability, the marginal of a joint distribution is obtained by summing over the discarded variable. In quantum mechanics, the analogous operation is the partial trace, which sums over a basis of the discarded subsystem.

The crucial structural fact: **tracing out a subsystem of a pure entangled state generally gives a mixed state.** This is the structural origin of decoherence. Unitary evolution preserves purity at the level of the full system, but "purity" can be lost in the transition to the subsystem description whenever the full state is entangled. A pure quantum state globally can look maximally mixed locally — that's entanglement.

The reverse operation, **purification**, says that every mixed state can be viewed as the marginal of a pure state on a larger Hilbert space. Mixed states are precisely the image of pure bipartite states under the partial trace.

Structural Role

Reduced density operators are the load-bearing object of open-system quantum mechanics:

- **Local observers use reduced states.** Any observer confined to a subsystem describes the world with a reduced density operator. This is the structural origin of why the world can *look* classical (mixed) even though the global wavefunction is pure.
- **Entanglement detection.** If $|\Psi\rangle_{AB}$ is pure, then $|\Psi\rangle$ is entangled iff ρ_A is mixed. Purity of the reduced state is the single cleanest test of pure-state entanglement.
- **Decoherence (Nodes 5.3–5.5).** Decoherence is formalized as the reduced dynamics of a system coupled to an environment: $\rho_S(t) = \mathrm{Tr}_E\big(U(t)(\rho_S(0) \otimes \rho_E)U^\dagger(t)\big)$. The map $\rho_S(0) \to \rho_S(t)$ is the open-system evolution.
- **Channels (Node 7.3).** Every quantum channel is a partial trace of a unitary on a larger space (Stinespring dilation). The partial trace is the structural universal building block of "generalized quantum dynamics."
- **Purification.** Every mixed state ρ_A can be written as $\mathrm{Tr}_B |\Psi\rangle\langle\Psi|$ for some pure $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. This embedding is one of the main tools of quantum information theory: to study a mixed state, lift it to a pure state on a larger space and work there.
- **Entanglement entropy (Node 4.5).** $S(\rho_A)$ of the reduced state is the canonical pure-state entanglement measure.

Mathematical Development

Linearity and basis-independence

The partial trace is a linear map $\mathrm{Tr}_B : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ that preserves Hermiticity, positivity, and trace. Basis-independence follows from the cyclic property: for two bases $\{|j_B\rangle\}, \{|j'_B\rangle\}$ related by a unitary U_B, the sums

$$\sum_j (I \otimes \langle j_B|)\rho(I \otimes |j_B\rangle) = \sum_{j,k,k'} U^*_{jk} U_{jk'} (I \otimes \langle k'_B|)\rho(I \otimes |k_B\rangle) = \sum_k (I \otimes \langle k_B|)\rho(I \otimes |k_B\rangle)$$

agree, using $\sum_j U^*_{jk} U_{jk'} = \delta_{kk'}$. The result depends only on the subsystem, not the choice of basis.

Schmidt decomposition and the spectrum of ρ_A

For a *pure* bipartite state $|\Psi\rangle_{AB} = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$ in Schmidt form (Node 4.3),

$$\rho_A = \mathrm{Tr}_B\,|\Psi\rangle\langle\Psi| = \sum_i \lambda_i^2\,|i_A\rangle\langle i_A|, \qquad \rho_B = \sum_i \lambda_i^2\,|i_B\rangle\langle i_B|.$$

Both reduced states are automatically diagonal in the Schmidt basis of their respective sides, and they share the same nonzero eigenvalues (the squared Schmidt coefficients). This is why entanglement entropy $S(\rho_A) = S(\rho_B)$ for pure bipartite states — the two marginals carry the same spectral information.

Purification

Theorem. Every density operator ρ_A on $\mathcal{H}_A$ is the reduced state of a pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ for some auxiliary space $\mathcal{H}_B$ with $\dim \mathcal{H}_B \geq \mathrm{rank}\,\rho_A$.

Construction. Diagonalize $\rho_A = \sum_i p_i |i\rangle\langle i|$. Take $\mathcal{H}_B$ with orthonormal basis $\{|i\rangle_B\}$ of the same dimension as the support of ρ_A, and set

$$|\Psi\rangle := \sum_i \sqrt{p_i}\,|i\rangle_A \otimes |i\rangle_B.$$

Then $\mathrm{Tr}_B\,|\Psi\rangle\langle\Psi| = \sum_i p_i |i\rangle_A\langle i|_A = \rho_A$. ■

Purifications are unique up to isometry on the auxiliary space $\mathcal{H}_B$, and the minimal purification has $\dim \mathcal{H}_B = \mathrm{rank}\,\rho_A$.

Why the diagonal-in-B prescription works

The formula $\mathrm{Tr}_B(\rho) = \sum_j \langle j_B|\rho|j_B\rangle_B$ (with implicit identity on A) may look ad hoc. The structural reason: it is the *unique* linear map satisfying $\mathrm{Tr}_B(A \otimes B) = A\,\mathrm{Tr}(B)$, and this in turn is forced by the operational requirement that expectation values of A-only observables be preserved. In category-theoretic language, the partial trace is the unique natural transformation making the corresponding diagram of C^*-algebras commute.

Reduced dynamics

If the joint state evolves unitarily under $U_{AB}(t)$, the reduced state on A evolves as

$$\rho_A(t) = \mathrm{Tr}_B\big(U_{AB}(t)\,(\rho_A(0) \otimes \rho_B(0))\,U_{AB}(t)^\dagger\big),$$

where we assumed an initially factorized state. This map $\rho_A(0) \mapsto \rho_A(t)$ is a **quantum channel** — completely positive and trace-preserving — but it is generally *not unitary*. The non-unitarity is the structural content of decoherence: the system's state evolution, in isolation, has lost information.

Worked Example

One qubit of a Bell pair

Take $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The density matrix of the joint state is

$$|\Phi^+\rangle\langle\Phi^+| = \tfrac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|).$$

Trace out qubit B:

$$\rho_A = \tfrac{1}{2}(|0\rangle\langle 0| \operatorname{Tr}|0\rangle\langle 0| + |0\rangle\langle 1| \operatorname{Tr}|0\rangle\langle 1| + |1\rangle\langle 0| \operatorname{Tr}|1\rangle\langle 0| + |1\rangle\langle 1| \operatorname{Tr}|1\rangle\langle 1|),$$

$$= \tfrac{1}{2}(|0\rangle\langle 0| \cdot 1 + |0\rangle\langle 1| \cdot 0 + |1\rangle\langle 0| \cdot 0 + |1\rangle\langle 1| \cdot 1) = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \tfrac{I}{2}.$$

The reduced state of one qubit of a Bell pair is **maximally mixed** — a completely uninformative state, indistinguishable from a fair coin flip between $|0\rangle$ and $|1\rangle$. All of the original pure state's information is stored in the *correlation* between the two qubits, not in either qubit alone.

This is the cleanest structural illustration of decoherence: a pure global state can look maximally mixed locally.

Reduced state of a partially entangled pure state

Take $|\Psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$. By the same calculation,

$$\rho_A = \cos^2\theta\, |0\rangle\langle 0| + \sin^2\theta\, |1\rangle\langle 1|.$$

The purity is $\operatorname{Tr}\rho_A^2 = \cos^4\theta + \sin^4\theta$, which ranges from 1 (at $\theta = 0$ or $\pi/2$, product state) to 1/2 (at $\theta = \pi/4$, Bell state). The entanglement entropy $h(\cos^2\theta)$ (binary entropy) behaves similarly.

Reduced state of a product state

For a product state $\rho_{AB} = \rho_A \otimes \rho_B$, the partial trace trivially gives back ρ_A:

$$\operatorname{Tr}_B(\rho_A \otimes \rho_B) = \rho_A \operatorname{Tr}(\rho_B) = \rho_A \cdot 1 = \rho_A.$$

No information is lost in the reduction, because ρ_B carries nothing that depends on A. Purity is preserved: if ρ_{AB} is a product of pure states, so is the reduction.

Classically correlated state

Consider $\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, a classically correlated mixed state. Tracing out B:

$$\rho_A = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \tfrac{I}{2}.$$

The reduced state is again $I/2$, *exactly the same* as the reduced state of $|\Phi^+\rangle$. From the marginal perspective, the Bell state and the classically correlated state are indistinguishable — the distinction lies entirely in the *correlations*, which are washed out by the partial trace. This is why entanglement cannot be detected from reduced states alone: you need access to cross-correlators like $\langle X_A X_B\rangle$.

Cross-Links

- **Module 4.3 (Schmidt decomposition):** Reduced states of bipartite pure states are diagonal in the Schmidt basis.
- **Module 4.5 (Entanglement measures):** The von Neumann entropy of ρ_A is the canonical pure-state entanglement measure.
- **Module 5.3 (System–environment coupling):** Reduced dynamics are defined by tracing out an unobserved environment.
- **Module 7.3 (Quantum channels):** Every CPTP channel arises as a partial trace of a unitary on a dilated space (Stinespring).

- **Statistics:** Marginalization of joint probability distributions. The partial trace generalizes $p(x) = \sum_y p(x, y)$ to non-commuting marginals.

Structural Compression

Invariant: ρ_A is the unique density operator on $\mathcal{H}_A$ such that $\mathrm{Tr}(\rho_A M_A) = \mathrm{Tr}(\rho_{AB}(M_A \otimes I_B))$ for every A-only observable M_A.

Mechanism: Sum over an orthonormal basis of $\mathcal{H}_B$: $\rho_A = \sum_j \langle j_B | \rho_{AB} | j_B \rangle$.

Cross-System Echo: Marginalization in classical probability integrates out the unobserved variable to give the marginal density. The partial trace is the non-commutative generalization — the operational restriction of a joint state to a subsystem that has no access to the rest.

Exercises

1. Verify directly that the partial trace of a density matrix is a density matrix: positive, Hermitian, unit-trace.
2. Show that Tr_B is independent of the basis chosen for $\mathcal{H}_B$.
3. Compute the reduced state of qubit A for each of the four Bell states. Verify that all four give $\rho_A = I/2$.
4. For $|\Psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, compute the purity of ρ_A as a function of θ and plot it.
5. Given the mixed state $\rho_A = 0.7|0\rangle\langle 0| + 0.3|1\rangle\langle 1|$, construct an explicit purification in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and verify that the reduced state on A matches.
6. Show that for a pure bipartite state, ρ_A is pure iff $|\Psi\rangle_{AB}$ is a product state.
7. Argue, structurally, why reduced density operators are the right objects for describing open quantum systems — what goes wrong if you try to use pure states instead?

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Concepts: density-matrix, tensor-product, entanglement
Prerequisites: 5.1, 4.1, 4.3
Enables: 5.3, 7.3

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