System–Environment Coupling

Quantum Foundations · Module 5 — Decoherence

Node 5.3

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Decoherence **Node:** 5.3

Every real quantum system is coupled to an environment. Photons, phonons, stray electromagnetic fields, nuclear spin baths, vibrational modes — no laboratory degree of freedom is truly isolated. **Decoherence** is the name for what happens when this coupling is taken seriously: the system's reduced dynamics become non-unitary, coherences between pointer states decay, and the quantum-mechanical description at the level of the system alone acquires classical features. This node sets up the general framework — total Hamiltonian $H = H_S + H_E + H_{SE}$, initial product state, unitary joint evolution, partial trace over the environment — that will be used throughout Modules 5 and 8.

Formal Definition

Consider a **system** S with Hilbert space $\mathcal{H}_S$ and an **environment** E with Hilbert space $\mathcal{H}_E$. The total Hilbert space is $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, and the total Hamiltonian decomposes as

$$H = H_S \otimes I_E + I_S \otimes H_E + H_{SE},$$

where H_S acts on the system alone, H_E on the environment alone, and H_{SE} is the interaction (generally non-factorizable).

Initial condition. At $t = 0$ we assume the joint state is factorized:

$$\rho_{\text{tot}}(0) = \rho_S(0) \otimes \rho_E(0).$$

This is an idealization (in realistic situations the system and environment have been coupled for some time prior to $t = 0$ and are not factorizable), but it is the starting point for the Born–Markov derivations of master equations (Node 8.2).

Joint unitary evolution. The total state evolves unitarily:

$$\rho_{\text{tot}}(t) = U(t)\rho_{\text{tot}}(0)U(t)^\dagger, \qquad U(t) = e^{-iHt/\hbar}.$$

Reduced system dynamics. The system state at time t is obtained by tracing out the environment:

$$\rho_S(t) := \mathrm{Tr}_E\big(U(t)[\rho_S(0) \otimes \rho_E(0)]U(t)^\dagger\big).$$

The map $\Phi_t : \rho_S(0) \mapsto \rho_S(t)$ is a **quantum channel** (completely positive, trace-preserving), but is generally **not unitary**. The non-unitarity is the mathematical signature of decoherence.

Intuition

The closed-system postulate (Node 2.4) said that quantum states evolve unitarily under the Schrödinger equation. This remains true for the *total* system + environment — the joint state is always pure, always evolves unitarily. What breaks is the assumption that the system alone has unitary dynamics. Once you couple the system to degrees of freedom you cannot observe, the reduced state stops behaving like a closed-system state.

The structural picture: imagine the system and environment are two dancers. Before the music starts they are independent; the state is a product $\rho_S \otimes \rho_E$. Once the interaction turns on they become entangled — the total state is no longer a product, even though it is still pure. If you watch only one dancer (the system), their motion becomes *correlated with* the other dancer's motion, and from the perspective of the isolated observation their behavior becomes stochastic. The "randomness" is not fundamental; it is a reflection of the unobserved correlations with the environment.

Operationally, decoherence is the process by which information *leaks* from system to environment. Phases of the system wavefunction that depend on which state the system is in get copied into environmental degrees of freedom, and once they are copied, coherent interference between those states becomes impossible from the system's perspective — the environment "knows which branch" and averaging over it destroys the off-diagonal terms.

The structural slogan: *The environment is an unasked-for measurement apparatus.* Its coupling to the system amounts to a continuous weak measurement in whatever basis the coupling selects, and the reduced system dynamics are exactly the measurement-induced dephasing of Node 3.6, averaged over an unread measurement record.

Structural Role

System–environment coupling is the universal frame for open quantum dynamics:

- **Decoherence (Nodes 5.4–5.6).** The specific reduced dynamics obtained by tracing out different kinds of environments — photon baths, phonon baths, spin baths — give the canonical dephasing and relaxation channels.
- **Pointer basis (Node 5.5).** The basis in which H_{SE} is diagonal is the one the environment "measures." It becomes the pointer basis of the decoherence process.
- **Open-system dynamics (Module 8).** The Lindblad master equation is the Markovian approximation of the exact reduced dynamics derived here. The non-Markovian corrections are the structural content of Node 8.5.
- **Decoherence-free subspaces.** A subspace of $\mathcal{H}_S$ that is invariant under H_{SE} does not decohere — it is protected from the environment by symmetry. This is the structural design principle behind several error-correcting codes and quantum memory architectures.
- **Stinespring / Naimark (Nodes 3.3, 7.3).** The "unitary on a larger space + partial trace" structure is the universal form of generalized quantum dynamics. System–environment coupling is the physical instantiation.

Mathematical Development

The reduced dynamics is a channel

Theorem. Given a factorized initial state $\rho_{\text{tot}}(0) = \rho_S(0) \otimes \rho_E(0)$ with fixed $\rho_E(0)$, the map $\Phi_t : \rho_S(0) \mapsto \rho_S(t)$ is completely positive and trace-preserving.

Proof sketch. Write the initial environment state in its spectral decomposition $\rho_E(0) = \sum_\mu p_\mu |e_\mu\rangle\langle e_\mu|$. Then

$$\rho_S(t) = \sum_\mu p_\mu \sum_\nu \langle e_\nu|U(t)|e_\mu\rangle \rho_S(0) \langle e_\mu|U(t)^\dagger|e_\nu\rangle = \sum_{\mu,\nu} K_{\mu\nu}(t)\rho_S(0)K_{\mu\nu}(t)^\dagger,$$

where $K_{\mu\nu}(t) := \sqrt{p_\mu}\langle e_\nu|U(t)|e_\mu\rangle$ are operators on $\mathcal{H}_S$. The completeness relation

$$\sum_{\mu,\nu} K_{\mu\nu}(t)^\dagger K_{\mu\nu}(t) = \sum_\mu p_\mu \sum_\nu \langle e_\mu|U(t)^\dagger|e_\nu\rangle\langle e_\nu|U(t)|e_\mu\rangle = \sum_\mu p_\mu \langle e_\mu|I|e_\mu\rangle = 1$$

confirms trace preservation. The Kraus representation makes complete positivity manifest. ■

Non-uniqueness of the Kraus decomposition

Different environment-basis choices give different Kraus operators, but the same channel Φ_t. This is the measurement-theoretic content of the Naimark dilation (Node 3.3): the physical channel is intrinsic to the joint unitary and the initial environment state, but its Kraus decomposition is basis-dependent.

Failure of divisibility

For general H_{SE}, the family of channels $\{\Phi_t\}_{t\geq 0}$ is *not* a one-parameter semigroup — that is, $\Phi_{t+s} \neq \Phi_t \circ \Phi_s$ in general. This is the structural reason that **non-Markovian** dynamics exist: information can flow from environment back to system, and the reduced dynamics carry memory. Markovian approximations (Node 8.2) are precisely those that discard this memory and treat $\{\Phi_t\}$ as a semigroup.

The Born approximation

In many physical situations the environment is vastly larger than the system and its state is barely perturbed by the interaction. The **Born approximation** treats $\rho_E(t) \approx \rho_E(0)$ throughout the evolution, and writes $\rho_{\text{tot}}(t) \approx \rho_S(t) \otimes \rho_E(0)$. Combined with a short-memory (Markov) assumption, this yields the Lindblad master equation of Node 8.2. The structural content of the Born approximation is: *the environment is much larger than the system and absorbs perturbations without reacting.*

Weak coupling and linear-response form

For $H_{SE} = g\, S \otimes E$ (a single system operator S coupled to a single environment operator E with strength g), second-order perturbation theory gives the leading decoherence rate

$$\frac{d\rho_S}{dt} \sim -g^2 \int_0^\infty d\tau\, C_E(\tau)[S,[S(-\tau),\rho_S]],$$

where $C_E(\tau) = \mathrm{Tr}(\rho_E E(\tau)E(0))$ is the environment's autocorrelation function. The decoherence rate is proportional to the environment's response function at the system's characteristic frequencies; the functional form of C_E determines whether the dynamics are Markovian (short correlations) or memory-laden (long correlations).

Worked Example

Single qubit dephasing via a bosonic bath

Take a qubit system with $H_S = \frac{\omega_0}{2}Z$ coupled to a bosonic environment (infinite set of harmonic oscillators) via

$$H_{SE} = Z \otimes \sum_k (g_k a_k^\dagger + g_k^* a_k),$$

and $H_E = \sum_k \omega_k a_k^\dagger a_k$. The total Hamiltonian is the **spin–boson model in the dephasing limit** — the coupling is diagonal in Z, so there is no energy exchange between system and environment, and the qubit's populations ($|0\rangle$ and $|1\rangle$ probabilities) are conserved. Only the phase relationship between them is affected.

Exact solution. The coupling is diagonal in Z, so the total evolution can be computed exactly. The off-diagonal matrix element of $\rho_S(t)$ evolves as

$$\rho_{01}(t) = \rho_{01}(0)\, e^{-i\omega_0 t}\, e^{-\Gamma(t)},$$

with

$$\Gamma(t) = 4\int_0^\infty d\omega\, \frac{J(\omega)}{\omega^2}(1-\cos\omega t)\coth\frac{\beta\omega}{2},$$

where $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$ is the environment's **spectral density** and β is the inverse temperature. The diagonal elements (populations) are unchanged.

- **Ohmic spectral density** $J(\omega) = \eta\omega e^{-\omega/\omega_c}$: $\Gamma(t) \sim \eta \ln(\omega_c t)$ at long times — logarithmic decoherence at zero temperature, linear in t at finite temperature.
- **Super-Ohmic** $J(\omega) \propto \omega^s$, $s > 1$: $\Gamma(t)$ saturates — the decoherence is bounded and reversible in the long-time limit. Entanglement is protected.
- **Sub-Ohmic** $J(\omega) \propto \omega^s$, $s < 1$: $\Gamma(t)$ grows faster than linearly — decoherence is stronger than Markovian.

The spectral density therefore determines whether the dynamics are Markovian, super-decoherent, or decoherence-protected; the functional form of $J(\omega)$ is a basic design parameter of quantum hardware.

The short-time linear regime

At very short times $t \ll 1/\omega_c$, regardless of the spectral density,

$$\Gamma(t) \approx \gamma\, t^2, \qquad \gamma = 2\int_0^\infty d\omega\, J(\omega)\coth\frac{\beta\omega}{2},$$

giving a quadratic-in-time decay. This is the **Zeno regime**, where frequent measurement (even weak) can freeze the dynamics and suppress decoherence. At slightly longer times $t \gg 1/\omega_c$ but still within the Markov regime, the decay becomes exponential: $\rho_{01}(t) \sim e^{-\Gamma_\infty t}$. The crossover is a structural feature of the coupling and is the reason Markovian approximations fail at short times.

The spin–boson model in the relaxing case

If instead the coupling were $H_{SE} = X \otimes \sum_k (g_k a_k^\dagger + g_k^* a_k)$ — a σ_x-type transverse coupling — the environment exchanges energy with the qubit, causing *relaxation* (amplitude damping) in addition to dephasing. The populations ρ_{00}, ρ_{11} now evolve toward thermal equilibrium, not just the off-diagonal elements. This is the structural distinction between T_2 **processes** (pure dephasing) and T_1 **processes** (relaxation), made precise in Node 5.4 and quantified in Node 5.6.

Cross-Links

- **Module 2.4 (Unitary evolution):** Closed-system dynamics are unitary; open-system dynamics are the partial trace of unitary dynamics on a larger space.
- **Module 3.6 (Back-action):** Decoherence is the back-action of the environment on the system averaged over an unread measurement.

- **Module 5.4 (Decoherence mechanisms):** Canonical channels (dephasing, amplitude damping) are the reduced dynamics in specific limits of system–environment coupling.
- **Module 8 (Open systems):** The Lindblad master equation is the Markovian approximation of the reduced dynamics derived here.
- **Module 7.3 (Stinespring dilation):** The structure "joint unitary + partial trace" is the universal form of CPTP maps; system–environment coupling is its physical realization.

Structural Compression

Invariant: The joint system-plus-environment state evolves unitarily; the reduced system state is obtained by partial trace. Reduced dynamics are in general non-unitary and non-Markovian.

Mechanism:

$$\rho_S(t) = \mathrm{Tr}_E\big(U(t)[\rho_S(0)\otimes\rho_E(0)]U(t)^\dagger\big)\,, \qquad U(t) = e^{-iHt/\hbar}, \quad H = H_S + H_E + H_{SE}.$$

Cross-System Echo: Coarse-graining in statistical mechanics — integrating out microscopic degrees of freedom to obtain effective macroscopic dynamics — is the same structural operation. Decoherence is the quantum-mechanical analogue of the second law's thermodynamic arrow: unitary global evolution, irreversible-looking local dynamics after tracing out the unobserved bath.

Exercises

1. Show that for a trivial interaction $H_{SE} = 0$, the reduced dynamics are unitary $\Phi_t(\rho_S) = e^{-iH_St/\hbar}\rho_S e^{iH_St/\hbar}$.
2. For $H_{SE} = gZ_S \otimes \sigma_z^E$ with a single-qubit environment initially in $(|0\rangle + |1\rangle)/\sqrt{2}$, compute the reduced system dynamics exactly and show that the off-diagonal element of ρ_S oscillates in time without decaying. Why doesn't a single-qubit environment cause decoherence?
3. For an N-qubit environment in $|+\rangle^{\otimes N}$ coupled as $H_{SE} = Z_S \otimes \sum_k \sigma_z^{(k)}$, compute $\rho_{01}(t)$ and show that it decays as $\cos^N(gt)$. Estimate how fast $N \to \infty$ causes decoherence.
4. Derive the short-time quadratic decay $\Gamma(t) \approx \gamma t^2$ for the spin–boson model with Ohmic spectral density.
5. Show that for a decoherence-free subspace $\mathcal{H}_{\mathrm{DFS}} \subseteq \mathcal{H}_S$ (a subspace on which H_{SE} acts as a multiple of the identity), states in the subspace evolve unitarily under H_S alone.
6. Compute the reduced dynamics for an initially entangled system–environment state $|\Psi_{SE}(0)\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ with $H_{SE} = 0$. Show that the "dynamics" look non-trivial even though the Hamiltonian is trivial. What goes wrong with the factorized-initial-state assumption?
7. Argue, structurally, why the factorized initial state $\rho_S \otimes \rho_E$ is an idealization, and what breaks down when the assumption fails.

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Concepts: unitary-evolution, density-matrix, decoherence
Prerequisites: 5.2, 2.4
Enables: 5.4, 8.1

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